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Probabilistic Selling in Quality-Differentiated Markets

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Probabilistic selling—the sale of synthetic products consisting of a lottery between two distinct goods—has been extensively analyzed in *horizontal* markets. In this research, we investigate probabilistic selling in *quality-differentiated* markets. This is an important new dimension of inquiry because of the widespread prevalence of quality-differentiated markets as well as significant differences in the preference structure across these markets. In fact, this latter consideration casts doubt as to whether probabilistic selling will even emerge in quality-differentiated markets. We find that probabilistic selling emerges in quality-differentiated markets as a way to profitably dispose excess capacity; moreover, probabilistic selling remains viable even under endogenous quality choice. In addition, in markets where sellers employ "strong" quality differentiation, the introduction of an intermediate probabilistic good actually causes *closer* quality levels in a product line and enhances consumer welfare. In contrast, in markets where sellers employ "weak" quality differentiation, the introduction of an intermediate probabilistic good increases quality separation and degrades consumer welfare. Overall, we view our contribution as one of characterizing the optimality, implementation, and policy implications of probabilistic selling in quality-differentiated markets.

Keywords: pricing; services marketing; probabilistic selling; quality choice; quality-differentiated markets; product line design; welfare
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1. Introduction

Consider the following observed market practices. An Internet broadband service provider offers two levels of service: Gold and Palladium. The Gold service is priced at $59.95 and offers a guaranteed download speed of 50 Mbps. The Palladium service, on the other hand, is priced lower at $49.95, but the download speed varies between 20 Mbps and 50 Mbps. Next, a major theme park offers two types of tickets: a higher-priced ticket that allows patrons to join a line with substantially reduced wait times and a lower-priced offering that is restricted to the regular line. This regular line may sometimes have reduced wait times and at other times suffer from long wait times. In a related context, a racquet club offers two levels of pricing for court time: a higher rate for guaranteed court availability and a lower rate that provides court time only as capacity becomes available. Again, the lower rate may sometimes yield a court immediately but at other times may involve a substantial wait. Finally, Hotwire, a travel intermediary, now lists the following menu with respect to rental cars: a compact car at $12.95/day or "special car" at $13.47/day. The special car includes the possibility of receiving a full-size car (which has a list price of $16.95/day) with the guarantee of receiving at least a compact car.

In all of these examples, one of the products in the product line essentially amounts to a lottery between two *quality-differentiated* goods. Indeed, the aforementioned emerging practices have been foreshadowed by many firms in the hotel and airline industries that have long offered upgrades to loyalty program members with some positive probability. In effect, these firms offer three products: premium (first class) at a high price, standard (economy) at a low price, and a premium upgrade subject to availability for those paying the loyalty membership fees. In fact, by now, there is a general understanding associated with the upgrade probability inherent in such offers. A travel website, Skyscanner.net, for example, reports a survey documenting that 42% of British Airways participants had received upgrades whereas only 1% of Iberia participants received upgrades (see Skyscanner 2010).

Following the nomenclature adopted by Fay and Xie (2008), we refer to the practice of offering a synthetic product consisting of a lottery between two distinct goods as *probabilistic selling*. Indeed, important aspects of probabilistic selling have been analyzed in Fay and

Xie (2008, 2010), Jerath et al. (2010), and Jiang (2007). However, these researchers focus on the use of probabilistic selling in *horizontal* markets, whereas we focus on the use of probabilistic selling in *quality-differentiated* markets. Empirically, although there are numerous horizontal markets where no preferred product exists (e.g., blue shirt versus red shirt or comedy versus action movie), there are equally numerous quality-differentiated markets where consumers strictly prefer one product over the other (a premium hotel room over a standard hotel room, a 50 Mbps Internet connection over 20 Mbps Internet connection, a full-size car over a compact car, etc.). Given the widespread incidence of quality-differentiated markets, it is important to ask whether probabilistic selling will prove profitable in quality-differentiated markets.

In addition to its managerial relevance, this question also has theoretical merit because there are important differences in the preference structure across horizontal and quality-differentiated markets. In horizontal markets, consumers in the middle are indifferent to the two extreme products. However, in quality-differentiated markets, there are no indifferent customers—all strictly prefer the product of higher quality. Accordingly, in horizontal markets, the introduction of probabilistic selling leads to higher price realization for the two deterministic products located at the extremes, as in Fay and Xie (2008). Alternatively, in Jerath et al. (2010), the introduction of probabilistic selling allows the seller to charge a higher price for the certain product in the first period. Here, the seller offers the probabilistic product with no surplus in the second period, but the consumer faces the risk that he or she will not receive the product if demand is high. In contrast, in quality-differentiated markets, the introduction of a probabilistic good lowers the price obtained for the high-quality offering on account of cannibalization. Given this degradation in price, it is not readily apparent whether probabilistic selling will enhance seller profits in such markets.

Against this backdrop, we first demonstrate the optimality of probabilistic selling in quality-differentiated markets. Specifically, we show that probabilistic selling emerges as a tool to profitably dispose excess capacity and obtain additional revenue from low-valuation customers. Crucially, the number of low-quality products included in the probabilistic offer can be suitably varied to mitigate the price degradation arising on account of cannibalization between the high-quality product and the probabilistic offer. Here, we also analyze the impact of transaction costs likely to emanate on account of including the probabilistic offer. For example, sellers may incur an additional cost to explicitly clarify the contract with customers who are purchasing the synthetic product. Sellers may also incur an additional fulfillment cost for probabilistic products on account of the somewhat more complex logistics and accounting associated with the probabilistic offer. Our analysis reveals that the impact of such transaction costs is to modify the product line offered by the seller because it reduces the ability of the seller to indiscriminately add low-quality products to reduce cannibalization.

Next, we recognize that an important feature of a quality-differentiated market is that it explicitly allows for quality choices. This is in contrast to horizontal markets, where the location of the product is generally fixed by researchers at the ends of the market. Given that sellers have this additional degree of freedom in quality-differentiated markets, the emergence of probabilistic selling under endogenous quality is again in doubt. In particular, it could well be that the suitable design of the product line can obviate the need for probabilistic selling. Moreover, the impact of introducing probabilistic selling on quality choices is interesting in its own right. Specifically, will the products come closer or move farther apart with the utilization of probabilistic selling? Similarly, with a view to understanding the policy implications associated with this emerging pricing format and quality choices, we also examine the impact of probabilistic selling on consumer surplus. Specifically, we ask the following: What is the impact of probabilistic selling on consumer surplus in quality-differentiated markets? Are consumers aided or hurt by this emerging pricing format?

Finally, we examine whether sellers will come to employ probabilistic selling in the face of demand uncertainty. Here, we demonstrate that probabilistic selling can increase seller profits when the taste for quality among low-type consumers is sufficiently high relative to their high-type counterparts. In this way, probabilistic selling emerges as a tool to manage adverse demand conditions.

In summary, the objectives of our research endeavor can be encapsulated by the following questions:

- Given differences in preference structures across horizontal and quality-differentiated markets, is probabilistic selling optimal in quality-differentiated markets? How does it improve profits relative to a world without probabilistic selling? What is the impact of transaction costs on the implementation of probabilistic selling?
- If sellers can choose quality in quality-differentiated markets, will they still come to utilize probabilistic selling? If yes, how are quality choices impacted by the introduction of probabilistic selling? Specifically, will the products come closer together or move farther apart with the introduction of probabilistic selling? Similarly, what is the impact of these choices on consumer surplus?
- Finally, can probabilistic selling be used as tool to manage adverse demand realizations?

Although probabilistic selling is a nascent pricing format, we believe that two developments portend its greater use: the worldwide growth in services and the increasing ability of technology to bind purchase and consumption. Both facilitate probabilistic selling because they limit the ability of customers to engage in arbitrage, which is a crucial requirement for probabilistic selling. Absent the ability to limit arbitrage, consumers (or entrepreneurial middlemen) can always remarket probabilistic products and thereby undo the basic segmentation scheme inherent in probabilistic selling. Such arbitrage is relatively difficult in the case of services because consumption requires the presence of the customer. Similarly, from a technological perspective, the use of services delivered electronically to an account (movies, music, etc.) allows consumption only from that account. As such, our research is particularly timely.

The rest of the paper is organized as follows. First, we briefly review the background literature and position our contributions relative to the extant work. Then, we present our model, analysis, and findings. Finally, we conclude with a discussion of our key findings and outline directions for future research.

2. Literature Review and Positioning

There is a growing body of research in marketing that analyzes the uncertainty inherent in market exchanges. The extant literature has highlighted two main aspects of uncertainty: *uncertainty in the buyer's consumption state* and *uncertainty in product offerings*. Uncertainty in the buyer's consumption state refers to the fact that buyers are often unsure about how much they value a future product. For example, buyers may be uncertain as to how strongly they will crave Chinese cuisine on some future occasion (Shugan and Xie 2000, Xie and Shugan 2001). In effect, this body of research introduces the notion of uncertainty in the buyer consumption state to reflect the fact that the utility obtained by buyers is likely to be influenced by various personal factors such as mood, work schedule, and family situation. These researchers then demonstrate the profit-enhancing ability of advanced selling. Such profit improvement arises because advanced selling allows the contract to be inked when both parties are equally uncertain about future consumption utilities.

In contrast, uncertainty in product offerings occurs when sellers engage in probabilistic selling. Here, the seller offers a good that essentially consists of a lottery between two distinct goods. Fay and Xie (2008) demonstrate how probabilistic selling can enhance profits via enhanced price discrimination and market expansion in horizontal markets. In a similar vein, Jerath et al. (2010) analyze whether a seller operating in a horizontal duopoly market should employ last-minute sales or choose to utilize probabilistic selling through an intermediary. There are several important differences between our work and that of Fay and Xie (2008) and Jerath et al. (2010). First, they focus on horizontal markets whereas we focus on quality-differentiated markets. As mentioned previously, the introduction of probabilistic selling in horizontal markets allows the seller to improve prices for the certain product, whereas in quality-differentiated markets, the introduction of probabilistic selling degrades the price realized for the high-quality product. This casts doubt on the optimality of probabilistic selling in quality-differentiated markets, thereby necessitating our formal inquiry. Second, in horizontal markets, the issue of product choice does not arise—the products are generally fixed by researchers to the ends of the market. However, in quality-differentiated markets, the seller can choose the qualities offered in the two segments. This difference gives rise to two additional research questions in our work: Will the freedom to choose quality obviate the need for probabilistic selling, and what is the impact of probabilistic selling on quality choices? Finally, we examine whether sellers will come to employ probabilistic selling in the face of demand uncertainty. We find that probabilistic selling may arise on account of demand uncertainty. Here, we are similar to Fay and Xie (2008) in that the seller must decide whether to use probabilistic selling before demand uncertainty is realized. This contrasts with Jerath et al. (2010), who analyze whether to use last-minute sales or an opaque intermediary after demand uncertainty is resolved.

In a related stream of work, Biyalagorsky et al. (2005) analyze probabilistic selling in the context of service upgrades where buyers of upgradeable tickets face a lottery between two classes of service. Although our work is closely related to their analysis, our problem context and results differ in the following manner. In their model, buyers of upgradeable tickets receive the higher class of service only if the high-type buyer fails to show up in the second period; moreover, this probability is *exogenous* to their analysis. In contrast, and as mentioned previously, the probability of receiving the high-quality product is a decision variable in our research setting. Furthermore, in their model, there is no degradation in price paid by the high-type consumer for the high-quality product on account of introducing probabilistic selling. This is because the two offers are temporally separate, and the high-type consumer only appears in the second period. However, in our research, as in Fay and Xie (2008), all consumers appear within the same period. As such, the introduction of probabilistic quality leads to cannibalization of the margin that the seller can obtain from sale of the high-quality product. These differences distinguish our work in important ways from the research of Biyalagorsky et al. (2005).

Finally, our paper also builds on the extant research on product line design. Deneckere and McAfee (1996) and Moorthy and Png (1992) focus on a quality-differentiated market characterized by two segments that differ in their taste for quality. They then analyze issues related to optimal product line design such as purposeful degradation of low quality below the efficient level to *reduce* cannibalization. Although we use many of their analysis techniques in our research, we differ primarily in that we introduce a synthetic third product that is a combination of the low- and high-quality products. Thus, in our research, we *increase* cannibalization by introducing a product that is closer to the high-quality product than the low-quality product. Nevertheless, we demonstrate how this practice can potentially enhance seller profits. In addition, we explicitly consider quality choices and examine how the utilization of probabilistic selling impacts quality choices. It is in this way that we complement the extant research on product line design.

3. Basic Model and Analysis

We begin by describing and analyzing a basic model where the quality choices of the seller are exogenous. In this context, we demonstrate the optimality of probabilistic selling in quality-differentiated markets. We also briefly describe the impact of transaction costs on the implementation of probabilistic selling.

3.1. Basic Model

Our model consists of a monopolist with two goods: a high-quality product and a low-quality product. In addition to these two quality-differentiated goods, the seller also has the option to include a synthetic product that essentially amounts to a lottery between the two goods. In this synthetic offer, the seller offers the high-quality good with some preannounced level of probability, ϕ. Correspondingly, this implies that the seller offers the low-quality good with probability $1-\phi$.

As described earlier, the seller may also face additional transaction costs stemming from clarification and fulfillment associated with offering a probabilistic product. We denote this additional transaction cost as c. It is incurred by the seller for each unit of the probabilistic product that is sold.

The seller has capacity M for the high-quality service and N for the low-quality service with $M < N$. This assumption is consistent with anecdotal evidence that sellers of two quality tiers generally offer more low-quality capacity. For example, the number of first-class seats on planes is typically lower than the number of economy seats. Similarly, hotels generally offer fewer premium suites relative to economy suites. Moreover, capacity choice is likely to be invariant in the short term; consequently, we exclude capacity choice in our research. With respect to costs, we set the associated variable costs for offering the high-quality and low-quality services at c_H and c_L, respectively, and without loss of generality, we assume $c_H \geq c_L$ with c_L normalized to zero. We next assume that there are two types of consumers in the market, *high-type* and *low-type*, with market sizes n_H and n_L, respectively, with $n_H < M$. Furthermore, the firm's optimization problem can be solved in two spaces: $n_L > N + M - n_H$ and $n_L \leq N + M - n_H$. We focus on the former space because it is likely to be more representative of real-world markets where the mass segment is generally large. However, as we show in the appendix, the emergence of probabilistic selling is robust to this consideration.

We further posit that the two types of consumers have different valuations for the two levels of services. The high-type consumers' value high quality at V_{HH} and low quality at V_{HL}, with $V_{HH} > V_{HL}$. The low-type consumers value high quality at V_{LH} and low quality at V_{LL}, with $V_{LH} > V_{LL}$. We also expect $V_{HH} > V_{LH}$ and $V_{HL} > V_{LL}$. Finally, we denote $[V_{HH} - V_{HL}] - [V_{LH} - V_{LL}] = [V_{HH} + V_{LL}] - [V_{HL} + V_{LH}]$ as Δ and expect $\Delta > 0$. That is, in addition to their stronger preference for a given level of quality, high-type consumers value successive levels of quality more than low-type consumers do. These assumptions are consistent with the basic quality-differentiated model presented in Tirole (1988, p. 296).

Within our model, consumers and sellers behave as follows. Consumers take their valuations as given and choose a service, with its associated price, so as to maximize utility. This utility consists of valuation for the service less the price charged by the seller. Sellers, on the other hand, take segments and valuations as given and offer products and set prices to maximize their profits.[1] In our model, we reiterate that the seller fixes the product line and pricing at the beginning of the period and all consumers appear randomly within the selling period. Furthermore, we make the tie-breaking assumption that if the consumer has to choose between two products with the same surplus, she buys the product favored by the buyer. Before we discuss our analysis, it is important to highlight that excess capacity is a necessary condition for probabilistic selling to arise in quality-differentiated markets. In other words, probabilistic selling will never arise without excess capacity. Formally, we state the following.

Lemma 1. *Probabilistic selling is never optimal if* $n_H \geq M$.

To facilitate exposition, the proof of Lemma 1, as well as all subsequent proofs, are relegated to the appendix.

[1] An additional assumption that we invoke pertaining to valuations is $V_{LH} - V_{LL} > C_H$. Since $\Delta > 0$, this implies $V_{HH} - V_{HL} > C_H$. These conditions imply that the seller obtains more profit by selling the high-quality product rather than the low-quality product to either type.

It is now useful to discuss three benchmarks that the seller has at his or her disposal that could be utilized in lieu of probabilistic selling.

3.2. Benchmarks

Here, we identify the profits for the seller without probabilistic selling. In this regard, there are three benchmarks that the seller can employ. In Benchmark 1, the seller focuses exclusively on serving the high-type customers with the high-quality product. This is an "up-market" strategy where the seller only offers the high-quality service at price V_{HH} to high-type consumers and excludes the low-type customers. Such an "exclusive" strategy allows the seller to obtain a high price, V_{HH}, for the high-quality product. The profits from this strategy are

$$p_H = V_{HH} \quad \Rightarrow \quad \pi_{B1} = n_H(V_{HH} - c_H). \tag{1}$$

Next, the seller offers a high-quality service to the high-type consumer and a low-quality service to low-type consumers, yielding the traditional differentiated product line strategy. We label this strategy Benchmark 2. Moreover, the seller charges V_{LL} for the low-quality product and $V_{HH} - V_{HL} + V_{LL}$ for the high-quality product. The price for the high-quality product, p_H, is obtained from the incentive-compatibility constraint such that the high-type consumer is indifferent to consuming the high- and low-quality products: $V_{HH} - p_H = V_{HL} - V_{LL}$. We call this as a "strong" differentiation strategy because the difference in price between the low and high quality is the difference in the *high-type's* valuations of these qualities subject to incentive compatibility. Prices and profits associated with Benchmark 2 can be formally stated as follows:

$$\begin{gathered} p_H = V_{HH} - V_{HL} + V_{LL} \quad \text{and} \\ p_L = V_{LL} \;\Rightarrow\; \pi_{B2} = n_H(V_{HH} + V_{LL} - V_{HL} - c_H) + NV_{LL}. \end{gathered} \tag{2}$$

In Benchmark 3, the entire capacity of the seller, $M + N$, is exhausted by setting a price of V_{LH} for the high-quality product and a price of V_{LL} for the low-quality product. In effect, the seller offers a high-quality service at the low-type consumers' reservation price V_{LH} and the low-quality service at the low-type consumers' reservation price V_{LL}. This also yields the traditional differentiated product line strategy, but we refer to this as a "weak" differentiation strategy. This is because the difference in price between the low and high quality is the difference in the *low-type's* valuations of these qualities. Here, the low-type consumers are indifferent between high- and low-quality goods. However, the high-type consumers will choose to buy the high-quality service because they obtain greater utility from consuming the high-quality service in this instance. Given more buyers than capacity, product assignment is random. Because seller profits are independent of the composition of buyers, prices and profits associated with Benchmark 3 are given as

$$\begin{gathered} p_H = V_{LH} \quad \text{and} \\ p_L = V_{LL} \quad \Rightarrow \quad \pi_{B3} = M(V_{LH} - c_H) + NV_{LL}. \end{gathered} \tag{3}$$

The three benchmarks vary in the price charged for the high-quality product. In the up-market strategy, the price obtained for the high-quality product is set to its highest level, namely, V_{HH}, because the low-quality product is not offered. Here, the seller is unconstrained while setting the price of the high-quality product. Next, in Benchmark 2, the price of the high-quality product is reduced to $V_{HH} + V_{LL} - V_{HL}$ because the seller offers both products and is constrained by incentive compatibility. Finally, in Benchmark 3, price is further reduced to V_{LH}. Here, the seller reduces the price to the valuation of the low-type consumer to ensure sufficient volume.

In summary, Benchmark 1 has a high price for high quality but excess capacity both for high quality and low quality. Benchmark 2 has a moderate price for the high-quality product but has some excess capacity for the high-quality product. Finally, Benchmark 3 has a relatively low price for the high-quality product but no excess capacity for either product. None of these benchmarks has a dominant advantage; consequently, all are worthy of retention for future analysis.

Before we proceed to our analysis, we must say a few words on when a particular benchmark may come to be preferred by a seller. If the market size of the high-type segment, n_H, or their valuation, V_{HH}, is high enough, the seller may want to ignore the low-type segment and only serve the high-type segment as described in Benchmark 1. However, if the low-type segment is substantially profitable relative to the high-type segment, then Benchmark 1 will cease to be optimal. In this event, the seller needs to make a choice between Benchmarks 2 and 3. In Benchmark 2, the seller tolerates unsold capacity of $M - n_H$ in order to obtain higher margins by targeting the high-quality product exclusively to the high-type segment. In contrast, Benchmark 3 allows for all of the high-quality capacity to be sold, albeit at a lower price. These considerations imply that when the high-type segment has a relatively high valuation or when $M - n_H$ is not large, Benchmark 2 makes sense. Otherwise, Benchmark 3 emerges as the optimal strategy for the seller.

3.3. Analysis

Benchmarks 1–3 illustrate the seller's options without adopting probabilistic selling in the product line. We next examine what the seller can do by employing probabilistic selling. Before we present this analysis, we preview how probabilistic selling can improve on the three benchmarks. Broadly speaking, probabilistic selling can improve on Benchmarks 1 and 2 by reducing excess capacity associated with these benchmarks. In contrast, probabilistic selling can improve on the relatively lower prices associated with Benchmark 3. Specifically, by including probabilistic selling, the seller can now target the high-quality product to the high-type segment and obtain a better price for the high quality while continuing to target the low-type segment with the probabilistic offer and low quality.

With some abuse of notation, we shall variously employ the symbol ϕ to reflect the probability of receiving high quality, a subscript to denote the level of the decision variable under probabilistic selling, or even the strategy of employing probabilistic selling in the product line. Note that when the probabilistic product is targeted to a particular segment, the value that buyers place on it is a linear combination of the valuations for high and low quality. That is, $V_{H\phi} = \phi V_{HH} + (1-\phi)V_{HL}$ and $V_{L\phi} = \phi V_{LH} + (1-\phi)V_{LL}$. Furthermore, the seller has to set prices with self-selection in mind, a point underscored in Moorthy (1984). That is, buyers will compute the utility of selecting every product offered by the seller and choose the product that maximizes their utility.

We first preview the manner of emergence of probabilistic selling. Whenever the firm chooses to offer a synthetic product, it is always targeted at the low-type segment. In these instances, the probabilistic-selling strategy can only take the following two forms:

(a) Probabilistic selling with *two* quality tiers $[H, \phi]$, where the seller offers n_H high-quality products to the high-type segment and cobbles the remaining $M - n_H$ high-quality products with N low-quality products to offer the synthetic product to the low-type segment.

(b) Probabilistic selling with *three* quality tiers $[H, \phi, L]$, where the seller offers n_H high-quality products to the high-type segment and offers the synthetic product and low-quality targeted to the low-type segment. In this event, all the remaining $M - n_H$ high-quality products and some of the N low-quality products are used to create the synthetic product. The remaining low-quality units are retained for separate sale to the low-type segment.

Given these considerations,

$$p_\phi = V_{L\phi} = \phi V_{LH} + (1-\phi)V_{LL}. \tag{4a}$$

In addition, to guarantee that the high-type consumers buy the high-quality product, we need

$$\begin{cases} V_{HH} - p_H \ge V_{H\phi} - p_\phi, \\ V_{HH} - p_H \ge V_{HL} - p_L, \end{cases}$$

$$\Rightarrow \begin{cases} p_H \le V_{HH} - \phi(V_{HH} - V_{LH}) - (1-\phi)(V_{HL} - V_{LL}), \\ p_H \le V_{HH} - (V_{HL} - V_{LL}). \end{cases}$$

Since $\Delta = (V_{HH} - V_{LH}) - (V_{HL} - V_{LL}) \ge 0$,

$$\begin{aligned} &V_{HH} - \phi(V_{HH} - V_{LH}) - (1-\phi)(V_{HL} - V_{LL}) \\ &\quad \le V_{HH} - (V_{HL} - V_{LL}) \\ &\quad \Rightarrow p_H = V_{HH} - \phi(V_{HH} - V_{LH}) - (1-\phi)(V_{HL} - V_{LL}) \\ &\qquad = V_{HH} - V_{HL} + V_{LL} - \phi\Delta. \end{aligned} \tag{4b}$$

Equation (4b) reveals that offering the synthetic product to the low-type consumer degrades the price that the seller can charge for the high-quality product offered to the high-type consumer. That is, the introduction of the synthetic product cannibalizes the margin that the seller can obtain from the high-quality product. Interestingly, the degradation in price is more pronounced as probabilistic quality becomes a closer substitute to the high-quality product (higher ϕ).

At this point, it is also useful to compare our quality-differentiated market structure to the horizontal structures analyzed in Fay and Xie (2008). The persistent difference is that we assume $V_{HL} > V_{LL}$, whereas Fay and Xie assume $V_{LL} > V_{HL}$. From Equation (4b), note that our assumption of $V_{HL} > V_{LL}$ in quality-differentiated markets degrades the price of the high-quality product more than the assumption of $V_{LL} > V_{HL}$ employed previously. This heightened cannibalization casts doubt as to whether probabilistic selling will even emerge in quality-differentiated markets, thereby motivating our formal inquiry.

To reiterate, offering probabilistic selling implies that the seller always targets high quality to the high-type consumers and then cobbles together the unsold high-quality capacity $M - n_H$ and some low-quality capacity X to create the synthetic product. When consumers buy this kind of product, they are not guaranteed to receive either the high or low quality; rather, consumers obtain the high-quality service with probability $\phi = (M - n_H)/(M - n_H + X)$ and low-quality product with probability $1 - \phi = X/(M - n_H + X)$.

Accordingly, in our analysis, a key decision variable is X, the amount of low-quality product that must be added to the excess high-quality capacity $M - n_H$ to create the synthetic product. Also, the cost for the synthetic product is $\phi c_H + c$, which includes both the product cost as well as the aforementioned transaction cost. Under these considerations, the seller chooses X to maximize the following profit function:

$$\begin{aligned} \pi &= n_H(p_H - c_H) \\ &\quad + (M - n_H + X)(p_\phi - \phi c_H - c) + (N - X)p_L \\ &= n_H(\Delta + c) + M(V_{LH} - c_H - c) + NV_{LL} \\ &\quad - cX - \left[\frac{(M - n_H)n_H}{M - n_H + X}\right]\Delta \quad \text{s.t. } 0 \le X \le N. \end{aligned} \tag{5}$$

3.4. Findings

We are now in a position to outline the propositions that flow from our analysis. Our first proposition pertains to the emergence and design of probabilistic selling in quality-differentiated markets. Formally, we have the following.

Proposition 1. *Sellers facing excess capacity in quality-differentiated markets will always find it profitable to employ probabilistic selling. Moreover, they will choose to offer it in one of two variants:* $[H, \phi, L]$ *or* $[H, \phi]$*, with associated prices, probability, units, and profits as detailed in Table* 1. *In addition, product line choice is determined by the magnitude of transactions costs: sellers facing relatively low transaction costs will choose the* $[H, \phi]$ *strategy, whereas sellers facing relatively high transaction costs will choose the* $[H, \phi, L]$ *strategy.*

(*To facilitate exposition, the somewhat lengthy expressions for all cost thresholds are detailed in Equations* (14) *and* (15) *in the appendix.*)

Proposition 1 reveals the optimality of probabilistic selling in quality-differentiated markets and the manner of its emergence. Of course, the emergence of probabilistic selling is conditional on the magnitude of the transaction costs. If this friction is too great, probabilistic selling ceases to be optimal. Proposition 1 underscores that probabilistic selling is useful in quality-differentiated markets when there is excess capacity.

We now provide some intuition for how probabilistic selling comes to dominate the benchmarks. First, we compare probabilistic selling with Benchmark 1. Note that, with the introduction of probabilistic selling, the price obtained for the high-quality product goes down in relation to the price obtained for the high-quality product in Benchmark 1 (cannibalization); however, the sales of unused high- and low-quality capacity to the low-type consumer enhances seller profits. Next, comparing probabilistic selling with Benchmark 2, the price obtained for the high-quality product goes down in relation to the price obtained for the high-quality product in Benchmark 2 (cannibalization). Specifically, in the $[H, \phi]$ offering, $p_H = \Delta + V_{LH} - (M - n_H)\Delta/(M + N - n_H)$, whereas in Benchmark 2, $p_H = \Delta + V_{LH}$. Similarly, in the $[H, \phi, L]$ offering, $p_H = \Delta + V_{LH} - \sqrt{c(M - n_H)\Delta/n_H}$, whereas in Benchmark 2, $p_H = \Delta + V_{LH}$. This cannibalization notwithstanding, the benefit of probabilistic selling is that it can gainfully utilize excess high-quality capacity.

Finally, we compare probabilistic selling to Benchmark 3. The advantage of probabilistic selling is that the seller is able to obtain a better price for the high-quality capacity by targeting the high-type customer. Specifically, in the $[H, \phi]$ offering, $p_H = \Delta + V_{LH} - (M - n_H)\Delta/(M + N - n_H)$, whereas in Benchmark 3, $p_H = V_{LH}$. It is straightforward to see that p_H in $[H, \phi]$ is always greater than $\mathbf{p}_H$ in Benchmark 3 since $\Delta > 0$. Similarly, in the $[H, \phi, L]$ offering, $p_H = \Delta + V_{LH} - \sqrt{c(M - n_H\Delta)/n_H} = \Delta + V_{LH} - \phi^*\Delta$, whereas in Benchmark 3, $p_H = V_{LH}$. Because ϕ^* is less than 1, it is straightforward to see that p_H in $[H, \phi, L]$ is also greater than p_H in Benchmark 3.

At this point, it is important to note that the probability associated with probabilistic selling implicitly controls the extent of the cannibalization effect. When ϕ is high, there is a greater proportion of high-quality product in the probabilistic offer; consequently, cannibalization is severe because the products are close substitutes. On the other hand, when ϕ is low (obtained by choosing a high X), it leads to a large number of units that are sold via probabilistic selling. Selling a large number of units via probabilistic selling is problematic for the seller because the transaction cost, c, applies to each probabilistic sale and erodes profits. Following this discussion, we can now understand why the choice of $[H, \phi]$ or $[H, \phi, L]$ is driven by the magnitude of the transaction costs. When transaction costs are absent or relatively low, the concern about profit erosion stemming from transaction costs is not particularly salient. Thus, it is optimal to minimize the cannibalization effect by exhausting all the low-quality product to create the probabilistic offer via the $[H, \phi]$ strategy. In contrast, as transaction costs increase, the

Table 1 **Prices, Probability, Units, and Profits Under** $[H, \phi]$ **and** $[H, \phi, L]$ **Strategies**

Strategy and decision variables	Level
Strategy $[H, \phi]$	
High-quality price	$p_H = \Delta + V_{LH} - \dfrac{(M - n_H)\Delta}{M + N - n_H}$
Low-quality price	$p_\phi = V_{LL} + \dfrac{(M - n_H)(V_{LH} - V_{LL})}{M + N - n_H}$
Unit	$X^* = N$
Probability	$\phi^* = \dfrac{M - n_H}{M - n_H + N}$
Profit	$\pi^*_{H\phi} = M(V_{LH} - c_H) + NV_{LL} + \dfrac{n_H N\Delta}{M + N - n_H} - c(M - n_H + N)$
Strategy $[H, \phi, L]$	
High-quality price	$p_H = \Delta + V_{LH} - \sqrt{\dfrac{c(M - n_H)\Delta}{n_H}}$
Low-quality price	$p_\phi = V_{LL} + (V_{LH} - V_{LL})\sqrt{\dfrac{c(M - n_H)\Delta}{n_H}}$
Unit	$X^* = \sqrt{\dfrac{(M - n_H)n_H\Delta}{c}} - (M - n_H)$
Probability	$\phi^* = \sqrt{\dfrac{c(M - n_H)}{n_H\Delta}}$
Profit	$\pi^*_{H\phi L} = n_H\Delta + M(V_{LH} - c_H) + NV_{LL} - 2\sqrt{c(M - n_H)n_H\Delta}$

concern about profit erosion stemming from transaction costs becomes salient. Consequently, the seller cannot *minimize* the cannibalization effect by indiscriminately adding a large number of low-quality products to create the probabilistic offer. As such, the seller does not exhaust the low-quality capacity while creating the probabilistic offer but rather reserves some low-quality product for separate sale via the $[H, \phi, L]$ strategy.

4. Endogenizing Quality

Here, we allow for quality choice. Our primary intent in this analysis is to understand whether freedom to choose quality levels can obviate the need for probabilistic selling in the face of excess capacity. We consider a scenario where the capacities continue to be fixed at M and N because capacity decisions are generally more long term in nature. Moreover, we further assume that it is not possible to shuffle a given capacity across qualities. That is, although the quality of each capacity type can be modified, infrastructure/fixed investments preclude more than one quality level being offered with each capacity type. Examples of this decision sequence include a hotel with a block of larger premium rooms and another block of smaller economy rooms or an airline with a designated first-class cabin on the top deck and a designated economy cabin in the lower deck. In both of these cases, the service provider has relatively more freedom to vary quality levels (via better amenities, meals, etc.) within a class but is relatively constrained with respect to capacities.

In this context, we demonstrate that endogenizing quality choice does not obviate the need for probabilistic selling. We also show how the introduction of probabilistic selling impacts quality choices. Finally, we examine the impact of utilizing probabilistic selling in quality-differentiated markets on consumer surplus.

4.1. Model

In our analysis here, we closely follow Moorthy and Png (1992). Accordingly, we posit that low-valuation customers value one unit of quality at v_L whereas high-valuation customers value one unit of quality at v_H, with $v_H > v_L$. In addition, the seller incurs cost αq^2, where q is the level of quality and α is a scaling parameter. If the seller offers quality q_L and q_H, then the corresponding valuations among the low- and high-valuation customers are $V_{LL} = v_L q_L$, $V_{LH} = v_L q_H$, $V_{HL} = v_H q_L$, and $V_{HH} = v_H q_H$. Similar to Moorthy and Png, when quality is chosen endogenously, the seller's optimal quality choices and the corresponding pricing and profits in each benchmark are shown below (detailed derivations are in the appendix).

Benchmark 1. Up-market strategy where the seller only offers the high-quality service to high-type consumers. The seller's quality choice in this benchmark is $q_H^* = v_H/(2\alpha)$ with corresponding prices $p_H^* = v_H^2/(2\alpha)$ and profit $\pi_{B1}^* = n_H v_H^2/(4\alpha)$.

Benchmark 2. Strong differentiation strategy where the seller offers high-quality service to the high-type consumer and low-quality service to low-type consumers. The seller's choices in this benchmark are

$$\begin{cases} q_L^* = \dfrac{v_L}{2\alpha} - \dfrac{n_H(v_H - v_L)}{2\alpha N}, \\ q_H^* = \dfrac{v_H}{2\alpha}, \end{cases}$$

with corresponding prices

$$\begin{cases} p_L^* = \dfrac{v_L^2}{2\alpha} - \dfrac{n_H(v_H - v_L)v_L}{2\alpha N}, \\ p_H^* = \dfrac{v_H^2}{2\alpha} - \dfrac{(v_H - v_L)v_L}{2\alpha} + \dfrac{n_H(v_H - v_L)^2}{2\alpha N}, \end{cases}$$

and profit $\pi_{B2}^* = n_H v_H^2/(4\alpha) + (1/(4\alpha N))[N v_L - n_H \cdot (v_H - v_L)]^2$. Consistent with Moorthy and Png (1992), the high-type consumers receive the efficient level of quality whereas the low-type receive a level of quality that is lower than the efficient level. Since quality q_L^* has to be nonnegative, Benchmark 2 will emerge only when $R \le 1$, where $R = n_H(v_H - v_L)/(N v_L)$ is the cannibalization index suggested by Moorthy and Png (1992).

Benchmark 3. Weak differentiation strategy where the seller offers both high- and low-quality services at the low-type consumers' reservation price. The seller's choices in this benchmark are

$$\begin{cases} q_L^* = \dfrac{v_L}{2\alpha}, \\ q_H^* = \dfrac{v_L}{2\alpha}, \end{cases}$$

with corresponding prices

$$\begin{cases} p_L^* = \dfrac{v_L^2}{2\alpha}, \\ p_H^* = \dfrac{v_L^2}{2\alpha}, \end{cases}$$

and profit $\pi_{B3}^* = (M + N)v_L^2/(4\alpha)$.

Under probabilistic selling, the seller's profit function is then obtained by substituting $V_{LL} = v_L q_L$, $V_{LH} = v_L q_H$, $V_{HL} = v_H q_L$, and $V_{HH} = v_H q_H$ in Equation (5) and incorporating the cost for quality. We obtain

$$\pi = \frac{N n_H}{M - n_H + N}(v_H - v_L)(q_H - q_L) + M(v_L q_H - \alpha q_H^2) + N(v_L q_L - \alpha q_L^2). \quad (6)$$

In Equation (6), and in our subsequent analysis, we suppress transaction costs ($c = 0$) to facilitate the exposition. This implies that the seller utilizes all the low-quality product to create the probabilistic offer via the $[H, \phi]$ strategy. This, in turn, implies that $X = N$. By solving, we obtain the seller's choices as

$$\begin{cases} q_L^* = \dfrac{v_L}{2\alpha} - \dfrac{n_H(v_H - v_L)}{2\alpha(M - n_H + N)}, \\ q_H^* = \dfrac{v_L}{2\alpha} + \dfrac{N n_H(v_H - v_L)}{2\alpha M(M - n_H + N)}, \end{cases}$$

with corresponding prices

$$\begin{cases} p_\phi^* = \dfrac{v_L}{2\alpha M(M - n_H + N)^2} \\ \qquad \cdot \{[(M - n_H)^2 + MN](M + N)v_L - n_H^2 N v_H\}, \\ p_H^* = \dfrac{n_H^2 N(v_H - v_L)^2}{2\alpha M(M - n_H + N)^2} + \dfrac{v_L^2}{2\alpha} + \dfrac{n_H N(v_H - v_L)v_L}{2\alpha M(M - n_H + N)}, \end{cases}$$

and profit $\pi_{PS}^* = (M+N)v_L^2/(4\alpha) + n_H^2(M+N)N(v_H - v_L)^2/(4\alpha M(M - n_H + N)^2)$.

Since quality q_L^* must be nonnegative, probabilistic selling will emerge only when $R \leq 1 + (M - n_H)/N$.

Next, by comparing probabilistic selling with the three benchmarks, we reveal conditions for the optimality of probabilistic selling strategy:

1. When $R > 1 + (M - n_H)/N$, both Benchmark 2 and the probabilistic-selling strategy cannot emerge. In addition, Benchmark 3 is dominated by Benchmark 1; consequently, the optimal strategy here is Benchmark 1.
2. When $1 < R \leq 1 + (M - n_H)/N$, Benchmark 2 cannot emerge, whereas the probabilistic-selling strategy can emerge. Here, we find that probabilistic selling is dominated by Benchmark 1; moreover, Benchmark 1 is again the optimal strategy.
3. When $R \leq 1$ (i.e., $v_L \geq n_H v_H/(n_H + N)$), both Benchmark 2 and the probabilistic-selling strategy are feasible. It is clear that Benchmark 1 is dominated by Benchmark 2 since $\pi_{B2}^* - \pi_{B1}^* = (1/(4\alpha N))[N v_L - n_H(v_H - v_L)]^2 > 0$. In the absence of probabilistic selling, the seller thus has two options: Benchmarks 2 or 3. When $v_L \leq (\sqrt{n_H(n_H + N)/N}\, v_H)/(\sqrt{n_H(n_H + N)/N} + \sqrt{M - n_H})$, Benchmark 2 is better than Benchmark 3, and when $v_L > \sqrt{n_H(n_H + N)/N}\, v_H/(\sqrt{n_H(n_H + N)/N} + \sqrt{M - n_H})$, Benchmark 3 dominates Benchmark 2.

First, consider $v_L \leq \sqrt{n_H(n_H + N)/N}\, v_H/(\sqrt{M - n_H} + \sqrt{n_H(n_H + N)/N})$. Here, Benchmark 2 is the relative benchmark. We now formally present our findings that delineate the seller's choices relative to Benchmark 2.

Proposition 2A. *When $v_L \leq \sqrt{n_H(n_H + N)/N}\, v_H/(\sqrt{n_H(n_H + N)/N} + \sqrt{M - n_H})$, probabilistic selling is optimal as long as the quality valuation by the low-type consumers exceeds a threshold: $v_L > \gamma_1 v_H$; otherwise, for $n_H v_H/(n_H + N) \leq v_L \leq \gamma_1 v_H$, Benchmark 2 will be the optimal strategy.*[2]

Proposition 2B. *Compared with Benchmark 2, utilization of probabilistic selling causes the seller to* lower *the quality and price of the high-quality product but* raise *the quality of the low-quality product.*

The intuition behind Propositions 2A and 2B is as follows. First, the willingness to pay for quality within the low-type segment should be above some threshold for probabilistic selling to emerge. If this willingness is too low, it leads to severe cannibalization. Moreover, a low willingness also implies that there is not much benefit to targeting the low-type consumer with the high-quality product in probabilistic selling. For these reasons, the emergence of probabilistic selling requires sufficiently high willingness to pay for quality among the low-type consumers.

Second, Proposition 2B reveals that the introduction of probabilistic selling brings the product line *closer* with respect to quality: the quality of the product targeted to the high-type segment is lowered while the quality of the product targeted to the low-type segment is raised. This is clearly surprising. One would expect that the introduction of an intermediate would be accompanied by greater separation between the two products. However, the seller chooses to do just the opposite. To understand the lowering of q_H^*, note that in the benchmark the high-quality product is targeted exclusively to high-type consumers and the optimal quality for the high-type consumer is at the efficient level, $q_H^* = v_H/(2\alpha)$. Now, when probabilistic quality is introduced, the total capacity of M high-quality products is sold partly in deterministic fashion to the high-type segment and partly via probabilistic selling to the low-type consumer. Because the low-type consumers have a lower taste for quality, the previous efficient level for the high-quality product $v_H/(2\alpha)$ is now too high. Consequently, given the ability to choose quality, the seller will reduce the quality of the high-quality product. Next, to understand the increase in q_L^*, note first that the low-quality product is targeted only to the low-type consumer; consequently, the customer base receiving this product remains unchanged. Now, recall that the reason the low quality was depressed below the efficient level in the benchmark was to reduce cannibalization. Specifically, the price for the

[2] Here,

$$\gamma_1 = \frac{\sqrt{\dfrac{M n_H(M - n_H + N)^2(n_H + N) - n_H^2(M + N)N^2}{MN(M - n_H + N)^2}}}{\sqrt{\dfrac{M n_H(M - n_H + N)^2(n_H + N) - n_H^2(M + N)N^2}{MN(M - n_H + N)^2}} + \sqrt{M - n_H}}$$

and $\dfrac{n_H v_H}{n_H + N} < \gamma_1 v_H < \dfrac{\sqrt{n_H(n_H + N)/N}\, v_H}{\sqrt{n_H(n_H + N)/N} + \sqrt{M - n_H}}$.

high-quality product in the benchmark is given as $p_H = V_{HH} - V_{HL} + V_{LL} = v_H q_H - (v_H - v_L)q_L$, where $(v_H - v_L)q_L$ is the reduction in price on account of cannibalization caused by the low-quality product. However, when probabilistic quality is introduced, $p_H = V_{HH} - V_{HL} + V_{LL} - \phi\Delta = v_H q_H - (1-\phi)(v_H - v_L)q_L - \phi(v_H - v_L)q_H$, where $(1-\phi)(v_H - v_L)q_L$ is the reduction in price on account of cannibalization caused by the low-quality product. Although overall cannibalization is higher with probabilistic selling, it is straightforward to observe that the cannibalization caused by the *low-quality* product within probabilistic selling is lower than in the benchmark (since $(1-\phi) < 1$). In addition, given that the number of low-type consumers served under probabilistic selling, $N + M - n_H$, is greater than those served in Benchmark 2, N, distortions in v_L become more costly. Both these considerations cause the seller to increase the quality of the low-quality product with a view to obtain greater profits from its sale.

Next, consider $v_L > \sqrt{n_H(n_H+N)/N}v_H)/(\sqrt{M-n_H} + \sqrt{n_H(n_H+N)/N})$. Here, Benchmark 3 is the relative benchmark. We now formally present our findings that delineate the seller's choices relative to Benchmark 3.

Proposition 3A. *When* $v_L > \sqrt{n_H(n_H+N)/N}v_H/(\sqrt{n_H(n_H+N)/N} + \sqrt{M-n_H})$, *probabilistic selling is always the optimal strategy.*

Proposition 3B. *Compared with Benchmark* 3, *utilization of probabilistic selling causes the seller to* raise *the quality and price of the high-quality product but* lower *the quality of the low-type product.*

Proposition 3A follows in straightforward fashion from the fact that $\pi^*_{PS} - \pi^*_{B3} = n_H^2(M+N)N \cdot (v_H - v_L)^2/(4\alpha M(M - n_H + N)^2) > 0$. The intuition behind Proposition 3B is also straightforward. Benchmark 3 is the weak differentiation strategy in which the quality gap between the high- and low-quality products is zero. Probabilistic selling improves on this extreme lack of differentiation by profitably separating the two products in terms of quality: the quality of high-quality product moves upward to enhance unit profitability, whereas the quality of low-quality product moves downward to avoid cannibalization.

In general, summarizing the results in Propositions 2A and 3A, the probabilistic-selling strategy is the optimal strategy as long as $v_L > \gamma_1 v_H$.

We now examine the impact of probabilistic selling on consumer surplus.

Proposition 4. *When quality is endogenous and the seller comes to utilize probabilistic selling, probabilistic selling improves consumer surplus relative to Benchmark* 2 *but lowers consumer surplus relative to Benchmark* 3.

Proposition 4 is understood as follows. Whenever surplus arises, it emerges only from the high-type segment through their consumption of the high-quality product because they never consume the probabilistic offer and the low-type segment is always kept at their valuation. Now, relative to Benchmark 2, the introduction of probabilistic selling lowers the quality of the high-quality product and raises the quality of the low-quality product. Recall that when probabilistic selling is adopted, $p_H = V_{HH} - V_{HL} + V_{LL} - \phi\Delta$. Consequently, the high-type consumers' surplus from buying the high-quality product is $V_{HL} - V_{LL} + \phi\Delta = (v_H - v_L)q_L + \phi(v_H - v_L)(q_H - q_L)$. In the benchmark, the high-type consumers' surplus is $V_{HL} - V_{LL} = (v_H - v_L)q_L$. The first term in the probabilistic-selling strategy, $(v_H - v_L)q_L$, is higher than the surplus in the benchmark even though they share the same expression, since q_L is higher in the probabilistic selling strategy relative to the benchmark. In addition, the second term, $\phi(v_H - v_L)(q_H - q_L)$, is positive. This term is induced by the utilization of probabilistic quality. Therefore, probabilistic selling always improves consumer surplus (as well as overall welfare because firms find it optimal to employ it). Intuitively, the introduction of probabilistic quality stimulates more price cannibalization of the high-quality product and consequently leads to higher surplus obtained by high-type consumers.

Next, in Benchmark 3, the consumer surplus obtained by the high-type consumers is $S_{B3} = n_H(v_H - v_L)q_H = n_H(v_H - v_L)v_L/(2\alpha)$, whereas the surplus in the probabilistic-selling strategy is $S_\phi = (n_H(v_H - v_L)v_L)/(2\alpha) - (n_H^3 N(v_H - v_L)^2)/(2\alpha M(M - n_H + N)^2)$. Clearly, S_{B3} is higher than S_ϕ because $S_{B3} - S_\phi = n_H^3 N(v_H - v_L)^2/(2\alpha M(M - n_H + N)^2) > 0$. Intuitively, both quality choice and the use of probabilistic selling affords the seller a variety of tools to extract more surplus from consumers as compared with offering undifferentiated products as in Benchmark 3.

5. Additional Consideration

Thus far, we have analyzed the seller's choice in a deterministic world. Here, we introduce demand uncertainty with a view to understanding whether probabilistic selling will continue to emerge if the seller has to determine quality levels and the utilization of probabilistic selling *before* demand uncertainty is resolved. To formally illustrate this conjecture, we focus on demand uncertainty with respect to n_H because it plays an important role within our model setting. In particular, we now posit that the seller faces low demand $n_{\underline{H}}$ with probability θ and high demand $n_{\bar{H}}$ with probability $1-\theta$, with $n_{\underline{H}} < M < n_{\bar{H}}$. In this scenario, the seller must decide whether to utilize probabilistic selling. Interestingly, we find out that probabilistic selling can continue to emerge when the cannibalization index $R' = [\theta n_{\underline{H}} + (1-\theta)M](v_H - v_L)/(N v_L)$ is sufficiently low. Again, Benchmarks 2 and 3 are the relative benchmarks under this condition. Moreover, probabilistic selling is the optimum as long as v_L is sufficiently high ($v_L > \gamma_2 v_H$).[3] In this event, the seller will reserve some high-quality capacity for probabilistic selling and cobble it together with some low-quality capacity to create the synthetic product. We state this formally below.

Proposition 5. *When the seller faces demand uncertainty of the form $n_{\underline{H}}$ with probability θ and $n_{\bar{H}}$ with probability $1-\theta$, $n_{\underline{H}} < M < n_{\bar{H}}$, probabilistic selling will still continue to emerge as the optimal strategy as long as $v_L > \gamma_2 v_H$.*

Intuitively, as summarized in Propositions 2A and 3A, the taste for quality among low-type consumers needs to be above a certain threshold for probabilistic selling to emerge. Note that Proposition 5 collapses to the joint results in Propositions 2A and 3A when $\theta = 1$, as it should.

6. Conclusion

Probabilistic selling is emerging as an important new pricing format. Whereas Fay and Xie (2008), Jerath et al. (2010), and Jiang (2007) investigate the role of probabilistic selling in horizontal markets, we focus on quality-differentiated markets. This is an important new dimension of inquiry because there are significant numbers of markets in which a quality-differentiated conceptualization is appropriate. Moreover, in horizontal markets, the seller is generally able to obtain higher prices for certain products under probabilistic selling. In contrast, in quality-differentiated markets, the seller is faced with degradation in price on account of cannibalization. This casts doubts as to whether this format will emerge in quality-differentiated markets and motivates our formal inquiry. Nevertheless, we first show that probabilistic selling can indeed emerge in quality-differentiated markets as a vehicle to profitably dispose excess capacity. Our results also reveal that the impact of increasing transaction costs is to alter the product line from $[H, \phi]$ to $[H, \phi, L]$.

Next, we endogenize the choice of quality. An interesting finding from our study is the emergence of probabilistic selling even when sellers have the freedom to choose quality. One could rightly conjecture that an appropriate design of the product line can obviate the need for probabilistic selling. However, this is not the case: probabilistic selling emerges even when sellers have the freedom to choose quality provided low-type consumers have a sufficiently high taste for quality. In addition, in markets where sellers employ quality differentiation (strong quality differentiation), the impact of introducing probabilistic selling on quality choices is to push the two qualities together despite the introduction of the intermediate, probabilistic good. Intuitively, one would expect the introduction of an intermediate product to effect greater separation among the extreme products. In this scenario, we also find that probabilistic selling can enhance consumer surplus. This effect arises even though high-quality consumers now come to receive a lower quality product. However, this is offset by a steeper decline in price on account of heightened cannibalization, leading to positive surplus. In contrast, in markets where sellers do not employ quality differentiation (weak quality differentiation), the impact of introducing probabilistic selling is to push the product line apart to take advantage of the benefits of differentiation. In addition, the ability to differentiate by quality and the utilization of the probabilistic offer provides the seller with a variety of tools to extract more consumer surplus; consequently, the introduction of probabilistic selling lowers consumer surplus here. Overall, the results pertaining to consumer surplus emulate the work of Jiang (2007) in horizontal markets—welfare may not always improve.

Finally, we examine whether sellers will come to employ probabilistic selling in the face of demand uncertainty. Here, we demonstrate that probabilistic selling can increase seller profits when the taste for quality among low-type consumers is sufficiently high relative to their high-type counterparts. In this way, probabilistic selling emerges as a tool to manage adverse demand conditions.

Of course, our work is not without limitations. Whereas this study is sharply focused on probabilistic selling in a quality-differentiated context to demonstrate its viability, we have omitted certain aspects that merit further attention. First, given the viability of probabilistic selling, it is possible that the monopolist can offer multiple tiers of probabilistic selling, with varying probabilities of obtaining the high-quality product. However, deciding on the number of tiers and the associated probabilities is not a simple task; thus, it is beyond the scope of the current work. Second, including temporal variation in willingness to pay (see, for example, Desiraju and Shugan 1999) and examining the impact of competition are also worthy of future research consideration.

[3] Here,

$$\gamma_2 = \left(\sqrt{\left[\frac{M[\theta n_{\underline{H}}+(1-\theta)M](M-n_{\underline{H}}+N)^2[[\theta n_{\underline{H}}+(1-\theta)M]+N]-n_{\underline{H}}^2(M+N)N^2}{MN(M-n_{\underline{H}}+N)^2}\right]}\right) \cdot \left(\sqrt{\left[\frac{M[\theta n_{\underline{H}}+(1-\theta)M](M-n_{\underline{H}}+N)^2[[\theta n_{\underline{H}}+(1-\theta)M]+N]-n_{\underline{H}}^2(M+N)N^2}{MN(M-n_{\underline{H}}+N)^2}\right]} + \sqrt{[M-[\theta n_{\underline{H}}+(1-\theta)M]]}\right)^{-1} < 1.$$

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Appendix

Proof of Lemma 1. *Probabilistic Selling Is Never Optimal When $n_H \geq M$.* To prove this, we compare the strategy in which the high-quality product is targeted to the high-type segment and low-quality product is targeted to the low-type segment with probabilistic selling. When utilizing probabilistic selling, the seller has to withhold some of the high quality sold directly to the high-type consumer and place it in the probabilistic offer. Since it is most profitable to sell the high-quality product directly to high-type consumers, the seller's profits strictly worsen by selling the high-quality product either to high- or low-type consumers via probabilistic selling. We thus conclude that the strategy in which the high-quality product is targeted to high-type consumers and low-quality product is targeted to low-type consumers dominates probabilistic selling, thereby proving that probabilistic selling will never emerge when $n_H \geq M$.

Proof of Proposition 1. *Manner of Emergence and Characterization of Probabilistic Selling*. Having shown that the seller will not target probabilistic quality to either the low- or high-type segment whenever $n_H \geq M$, we now focus on the case where $n_H < M$. Our goal here is to describe the manner in which probabilistic quality can emerge and thereby facilitate the subsequent exposition.

First, observe that there is no benefit in targeting probabilistic selling to the high-type segment. This is understood as follows. Consider the comparison of introducing probabilistic selling with Benchmark 2. Since $n_H < M$, if the seller wants to target the high-type segment with the probabilistic offer, then each high-type buyer of the probabilistic offer is precluded from buying the high-quality offering directly. Prices are obtained via incentive compatibility and are identical to those described in Equations (4a) and (4b). Accordingly, the profit from selling a unit of the probabilistic good is $p_\phi - \phi c_H - (1-\phi)c_L - c = \phi(V_{HH} - V_{HL} + V_{LL} - c_H) + (1-\phi)(V_{LL} - c_L) - c$, and the profit from selling a unit of high quality is $V_{HH} - V_{HL} + V_{LL} - c_H$. Since the latter exceeds the former by the positive quality, $(1-\phi)(V_{HH} - V_{HL} - c_H) + (1-\phi)c_L + c$, it is straightforward to conclude that the seller will not replace direct sales of high quality by probabilistic selling.

In contrast, targeting probabilistic selling to the low-type segment has the potential to (i) increase unit sales by serving the ignored low-type consumers in Benchmark 1, (ii) utilize the excess capacity in Benchmark 2, and (iii) increase price charged for the high-quality product sold to high-type consumers in Benchmark 3. Thus, probabilistic selling targeted to the low-type segment has the potential to increase profits.

Following this observation, we first show that $[\phi, L]$ is never optimal. In $[\phi, L]$ the seller does not offer the high-quality product to the market; the entire high-quality capacity is used to create the probabilistic offer. Suppose the seller uses Z low-quality products in addition to M high-quality products in the probabilistic offer. Thus, $\phi = M/(M+Z)$.

The seller's profit is

$$\begin{aligned}\pi &= (M+Z)[p_\phi - \phi c_H - (1-\phi)c_L - c] + (N-Z)(p_L - c_L)\\ &\Rightarrow \pi = (M+Z)\left[\frac{M(V_{LH}-V_{LL})}{M+Z} + V_{LL} - \frac{Mc_H}{M+Z} - \frac{Zc_L}{M+Z} - c\right] + (N-Z)(V_{LL} - c_L)\\ &\Rightarrow \pi = M(V_{LH} - c_H) + N(V_{LL} - c_L) - c(M+Z).\end{aligned}$$

This profit is lower than $\pi_{B3} = M(V_{LH} - c_H) + N(V_{LL} - c_L)$. Thus, $[\phi, L]$ is dominated by Benchmark 3.

Next, we examine the three-product solution: high-quality + probabilistic offer + low-quality (or $[H, \phi, L]$). Here, X low-quality units are sold with $M - n_H$ high-quality units together via probabilistic selling. The decision variable in the analysis is X. And n_H high-quality products are targeted to high-type consumers while the probabilistic product and low-quality product are sold to low-type consumers. When $n_L > M - n_H + N$, profits from offering probabilistic selling are given as

$$\begin{aligned}\pi &= n_H(p_H - c_H) + (M - n_H + X)[p_\phi - \phi c_H - (1-\phi)c_L - c]\\ &\quad + (N - X)(p_L - c_L)\\ &= n_H(\Delta + c) + M(V_{LH} - c_H - c) + N(V_{LL} - c_L) - cX\\ &\quad - \frac{(M-n_H)n_H}{M - n_H + X}\Delta. \end{aligned} \tag{7}$$

The first-order condition yields

$$\frac{\partial \pi}{\partial X} = 0 \Rightarrow X^* = \sqrt{\frac{(M-n_H)n_H\Delta}{c}} - (M - n_H) \tag{8}$$

$$\Rightarrow \phi^* = \sqrt{\frac{c(M-n_H)}{n_H\Delta}}. \tag{9}$$

Thus,

$$p_\phi^* = V_{LL} + (V_{LH} - V_{LL})\sqrt{\frac{c(M-n_H)}{n_H\Delta}} \tag{10}$$

and

$$\begin{aligned}\pi^* &= n_H\Delta + M(V_{LH} - c_H) + N(V_{LL} - c_L)\\ &\quad - 2\sqrt{c(M-n_H)n_H\Delta}.\end{aligned} \tag{11}$$

The second-order condition yields $\partial^2\pi/\partial X^2 = -2(M-n_H)\cdot n_H[(V_{HH}-V_{LH})-(V_{HL}-V_{LL})]/[(M-n_H)+X]^2 \leq 0$, since $\Delta = [(V_{HH}-V_{LH})-(V_{HL}-V_{LL})] > 0$. Hence, $X^* = \sqrt{(M-n_H)n_H\Delta/c} - (M-n_H)$ maximizes profit.

We now have two conditions involving X^* that shape the probabilistic offer:

1. $X^* > 0$: From the expression for X^*, $X^* > 0$ implies that $c < n_H\Delta/(M-n_H)$. (Note that this inequality, given (8), ensures that the probability associated with high-quality never exceeds 1.)
2. $X^* \leq N$: From the expression for X^*, $X^* \leq N$ implies that $c \geq (M-n_H)n_H\Delta/[(M-n_H)+N]^2$.

When $X^* > N$, there is not enough low-quality product to put in the probabilistic offer. The maximum amount of low-quality product that can be used is N. Thus, when $X^* > N$, the seller sets $X^* = N$. We thus have the two-segment solution $[H, \phi]$ instead of $[H, \phi, L]$, where the seller just offers the high-quality and the probabilistic product. The associated profit is thus obtained using the expressions for the prices, ϕ and X^* (X^* is now equal to N), in (8) to get

$$\begin{aligned}\pi_{H\phi}^* &= M(V_{LH} - c_H) + N(V_{LL} - c_L) + \frac{n_H N\Delta}{M + N - n_H}\\ &\quad - c(M - n_H + N).\end{aligned} \tag{12}$$

When $0 < X^* \le N$, the seller finds it optimal to offer probabilistic selling while setting some low-quality capacity aside for separate sale to the low-type segment, and we get the three-segment solution $[H, \phi, L]$. The condition $0 < X^* \le N$, given the expression for X^* from (8), translates to $(M-n_H)n_H\Delta/[(M-n_H)+N]^2 \le c < n_H\Delta/(M-n_H)$, and the associated profit is obtained by substituting the expressions for the prices and X^* in (12) to get

$$\pi^*_{H\phi L} = n_H\Delta + M(V_{LH}-c_H) + N(V_{LL}-c_L) - 2\sqrt{c(M-n_H)n_H\Delta}. \quad (13)$$

Optimal Solution: *Comparing Probabilistic Selling to Benchmarks* 1, 2, *and* 3. Comparing probabilistic selling to Benchmark 1, Benchmark 2, and Benchmark 3, we have, respectively,

$$\begin{cases} c < \dfrac{(M-n_H)n_H\Delta}{[(M-n_H)+N]^2} = c_1 \;\Rightarrow\; \pi^*_{H\phi} > \pi^*_{H\phi L}, \\ c < \dfrac{(M-n_H)(V_{LH}-c_H)-((M-n_H)n_H/((M-n_H)+N))\Delta}{(M-n_H)+N} \\ \quad = c_2 \;\Rightarrow\; \pi^*_{H\phi} > \pi_{B1}, \\ c < \dfrac{N_{n_H\Delta}}{[(M-n_H)+N]^2} = c_3 \;\Rightarrow\; \pi^*_{H\phi} > \pi_{B2}, \\ c < \dfrac{M(V_{LH}-c_H)-n_H(V_{HH}-c_H)+N(V_{LL}-c_L)}{(M-n_H)+N} \\ \quad + \dfrac{Nn_H\Delta}{[(M-n_H)+N]^2} = c_4 \;\Rightarrow\; \pi^*_{H\phi} > \pi_{B3}. \end{cases} \quad (14)$$

Thus, $\pi^*_{H\phi}$ is best when $c < \min(c_1, c_2, c_3, c_4)$.

In addition,

$$\begin{cases} c_1 \le c < \dfrac{n_H\Delta}{(M-n_H)} = c_5 \;\Rightarrow\; \pi^*_{H\phi L} > \pi^*_{H\phi}, \\ c < \dfrac{(M-n_H)(V_{LH}-c_H)^2}{4n_H\Delta} = c_6 \;\Rightarrow\; \pi^*_{H\phi L} > \pi_{B1}, \\ c < \dfrac{n_H\Delta}{4(M-n_H)} = c_7 \;\Rightarrow\; \pi^*_{H\phi L} > \pi_{B2}, \\ c < \dfrac{[(M-n_H)(V_{LH}-c_H)-n_H(V_{HL}-V_{LL})+N(V_{LL}-c_L)]^2}{4(M-n_H)n_H\Delta} \\ \quad = c_8 \;\Rightarrow\; \pi^*_{H\phi L} > \pi_{B3}. \end{cases} \quad (15)$$

Thus, $\pi^*_{H\phi L}$ is best when $c_1 \le c < \min(c_5, c_6, c_7, c_8)$.

Robustness Check: *Emergence of Probabilistic Selling When* $n_L \le M - n_H + N$. Here, we relax our assumption of $n_L > M - n_H + N$ and show that probabilistic selling can still emerge provided n_L is not too low. Thus, in this robustness check, we consider the range $M - n_H < n_L \le M - n_H + N$. We have

$$\begin{aligned} \pi &= n_H(p_H - c_H) + (M - n_H + X)[p_\phi - \phi c_H - (1-\phi)c_L - c] \\ &\quad + [n_L - (M - n_H + X)](p_L - c_L) \\ &= n_H(\Delta + c) + M(V_{LH} - c_H - c) - cX - \frac{(M-n_H)n_H}{M - n_H + X}\Delta \\ &\quad + [n_L - (M - n_H)](V_{LL} - c_L). \end{aligned}$$

The first-order condition yields

$$\frac{\partial \pi}{\partial X} = 0 \Rightarrow X^* = \sqrt{\frac{(M-n_H)n_H\Delta}{c}} - (M - n_H) \quad (16)$$

$$\Rightarrow \phi^* = \sqrt{\frac{c(M-n_H)}{n_H\Delta}}. \quad (17)$$

Thus,

$$p^*_\phi = V_{LL} + (V_{LH} - V_{LL})\sqrt{\frac{c(M-n_H)}{n_H\Delta}} \quad (18)$$

and

$$\pi^* = n_H\Delta + M(V_{LH} - c_H) - 2\sqrt{c(M-n_H)n_H\Delta} + [n_L - (M - n_H)](V_{LL} - c_L). \quad (19)$$

The second-order condition yields $\partial^2\pi/\partial X^2 = -(2(M-n_H) \cdot n_H[(V_{HH} - V_{LH}) - (V_{HL} - V_{LL})])/[(M-n_H) + X]^2 \le 0$, since $\Delta = [(V_{HH} - V_{LH}) - (V_{HL} - V_{LL})] > 0$. Hence, $X^* = \sqrt{(M-n_H)n_H\Delta/c} - (M - n_H)$ maximizes profit. Then we get $M - n_H + X^* = \sqrt{(M-n_H)n_H\Delta/c}$.

- If $c \ge (M-n_H)n_H\Delta/[(M-n_H)+N]^2$, then $\sqrt{(M-n_H)n_H\Delta/c} \le M - n_H + N$,

—when $\sqrt{(M-n_H)n_H\Delta/c} < n_L \le M - n_H + N$, the results in Equations (16)–(19) still hold;

—when $(M-n_H) < n_L \le \sqrt{(M-n_H)n_H\Delta/c}$, the market demand of low-type consumers is relatively low—the solution is at corner $M - n_H + X = n_L$ and is of the form $[H, \phi]$. We get $X^* = n_L - (M - n_H)$, $\phi^* = (M - n_H)/n_L$, and

$$\pi^*_{H\phi} = n_H\Delta + M(V_{LH} - c_H) - n_L c - \frac{(M-n_H)n_H}{n_L}\Delta + [n_L - (M - n_H)](V_{LL} - c_L).$$

- If $c < (M-n_H)n_H\Delta/[(M-n_H)+N]^2$, then $\sqrt{(M-n_H)n_H\Delta/c} > M - n_H + N$. So when $n_L \le M - n_H + N$, n_L is also smaller than $\sqrt{(M-n_H)n_H\Delta/c}$, which means the market demand of low-type consumers is relatively low, the optimal solution is at corner $M - n_H + X = n_L$ and of the form $[H, \phi]$. We get $X^* = n_L - (M - n_H)$, $\phi^* = (M-n_H)/n_L$, and $\pi^*_{H\phi} = n_H\Delta + M(V_{LH} - c_H) - n_L c - [(M-n_H)n_H/n_L]\Delta + [n_L - (M - n_H)](V_{LL} - c_L)$.

Proof of Propositions 2 and 3 (Endogenizing Quality Choice). For ease of exposition, here we set $c = 0$. As $c = 0$, the seller will assign the low-quality product to the synthetic product, therefore $X = N$.

q_H: Quality of high-quality product.
q_L: Quality of low-quality product.
v_H: High-type segment's taste for quality.
v_L: Low-type segment's taste for quality.
α: The cost coefficient. We assume a quadratic production cost, which implies that the cost of producing a product at quality level q is αq^2.

Benchmarks. First, we explore the profit of the three benchmarks above when the qualities of both high- and low-quality products are endogenized.

Benchmark 1. Target the high-quality product to the high-type segment only.

The price of the high-quality product is $p_H = V_{HH} = v_H q_H$. The seller's profit is

$$\pi_{B1} = n_H(p_H - c_H) = n_H(v_H q_H - \alpha q_H^2). \tag{20}$$

The first-order condition yields

$$\frac{\partial \pi_{B1}}{\partial q_H} = n_H(v_H - 2\alpha q_H) = 0 \;\Rightarrow\; q_H^* = \frac{v_H}{2\alpha} \tag{21}$$

and

$$\pi_{B1}^* = \frac{n_H v_H^2}{4\alpha}. \tag{22}$$

Benchmark 2. Target the high-quality product to the high-type segment and the low-quality product to the low-type segment.

The price for the low-quality product is $p_L = V_{LL} = v_L q_L$. This is obtained from the incentive-compatibility constraint such that the high-type consumer is indifferent between consuming the high- and low-quality products. We have $p_H = V_{HH} - V_{HL} + V_{LL} = v_H q_H - v_H q_L + v_L q_L$.

The seller's profit is

$$\begin{aligned}\pi_{B2} &= n_H(p_H - c_H) + N(p_L - c_L)\\ &= n_H(v_H q_H - v_H q_L + v_L q_L - \alpha q_H^2) + N(v_L q_L - \alpha q_L^2).\end{aligned} \tag{23}$$

The first-order condition yields

$$\frac{\partial \pi_{B2}}{\partial q_H} = n_H(v_H - 2\alpha q_H) = 0 \;\Rightarrow\; q_H^* = \frac{v_H}{2\alpha}, \tag{24}$$

$$\begin{aligned}\frac{\partial \pi_{B2}}{\partial q_L} &= -n_H(v_H - v_L) + N(v_L - 2\alpha q_L) = 0\\ &\Rightarrow q_L^* = \frac{v_L}{2\alpha} - \frac{n_H(v_H - v_L)}{2\alpha N}.\end{aligned} \tag{25}$$

Since q_L^* is nonnegative, we have $n_H(v_H - v_L)/(N v_L) \le 1$. It implies that $v_L \ge n_H v_H/(n_H + N)$. We define $R = n_H(v_H - v_L)/(N v_L)$ and conclude that Benchmark 2 will emerge only when $R \le 1$.

Thus,

$$\begin{aligned}p_L^* &= v_L q_L^* = \frac{v_L^2}{2\alpha} - \frac{n_H(v_H - v_L)v_L}{2\alpha N}, \quad \text{and}\\ p_H^* &= v_H q_H^* - v_H q_L^* + v_L q_L^*\\ &= \frac{v_H^2}{2\alpha} - \frac{(v_H - v_L)v_L}{2\alpha} + \frac{n_H(v_H - v_L)^2}{2\alpha N};\end{aligned} \tag{26}$$

and

$$\begin{aligned}\pi_{B2}^* &= \frac{n_H v_H^2}{4\alpha} + \frac{N v_L^2}{4\alpha} - \frac{n_H(v_H - v_L)v_L}{2\alpha} + \frac{n_H^2(v_H - v_L)^2}{4\alpha N}\\ &= \frac{n_H v_H^2}{4\alpha} + \frac{1}{4\alpha N}[N v_L - n_H(v_H - v_L)]^2.\end{aligned} \tag{27}$$

Benchmark 1 is dominated by Benchmark 2 when the qualities are endogenized.

Benchmark 3. Target the high-quality product to both the low- and high-type segments and the low-quality product to the low-type segment.

The price for the low-quality product is $p_L = V_{LL} = v_L q_L$, and the price for the high-quality product is $p_H = V_{LH} = v_L q_H$.

The seller's profit is

$$\begin{aligned}\pi_{B3} &= M(p_H - c_H) + N(p_L - c_L) = M(v_L q_H - \alpha q_H^2)\\ &\quad + N(v_L q_L - \alpha q_L^2).\end{aligned} \tag{28}$$

The first-order condition yields

$$\frac{\partial \pi_{B3}}{\partial q_H} = M(v_L - 2\alpha q_H) = 0 \;\Rightarrow\; q_H^* = \frac{v_L}{2\alpha}, \tag{29}$$

$$\frac{\partial \pi_{B3}}{\partial q_L} = N(v_L - 2\alpha q_L) = 0 \;\Rightarrow\; q_L^* = \frac{v_L}{2\alpha}, \tag{30}$$

and

$$\pi_{B3}^* = \frac{(M+N)v_L^2}{4\alpha}. \tag{31}$$

Probabilistic Quality Strategy. We replace V_{LL}, V_{LH}, V_{HL}, and V_{HH} in the seller's profit in Equation (7) in the basic model with $v_L q_L$, $v_L q_H$, $v_H q_L$, and $v_H q_H$. The seller's profit becomes

$$\begin{aligned}\pi_{PS} &= n_H(v_H - v_L)(q_H - q_L) + M(v_L q_H - \alpha q_H^2)\\ &\quad + N(v_L q_L - \alpha q_L^2) - \left[\frac{(M - n_H)n_H}{M - n_H + N}\right](v_H - v_L)(q_H - q_L)\\ &= \frac{N n_H}{M - n_H + N}(v_H - v_L)(q_H - q_L) + M(v_L q_H - \alpha q_H^2)\\ &\quad + N(v_L q_L - \alpha q_L^2).\end{aligned} \tag{32}$$

The first-order condition yields

$$\begin{aligned}\frac{\partial \pi_{PS}}{\partial q_H} &= \frac{N n_H}{M - n_H + N}(v_H - v_L) + M(v_L - 2\alpha q_H) = 0\\ &\Rightarrow q_H^* = \frac{v_L}{2\alpha} + \frac{N n_H(v_H - v_L)}{2\alpha M(M - n_H + N)} \quad \text{and}\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial \pi_{PS}}{\partial q_L} &= -\frac{N n_H}{M - n_H + N}(v_H - v_L) + N(v_L - 2\alpha q_L) = 0\\ &\Rightarrow q_L^* = \frac{v_L}{2\alpha} - \frac{n_H(v_H - v_L)}{2\alpha(M - n_H + N)}.\end{aligned} \tag{34}$$

Since q_L^* is nonnegative, we have $n_H(v_H - v_L)/(M - n_H + N)v_L \le 1$. It implies that $v_L \ge (n_H v_H)/(M+N)$. We conclude that the probabilistic-selling strategy will emerge when $R \le 1 + (M - n_H)/N$.

Thus,

$$\begin{aligned}p_H^* &= \frac{n_H^2 N(v_H - v_L)^2}{2\alpha M(M - n_H + N)^2} + \frac{v_L^2}{2\alpha}\\ &\quad + \frac{n_H N(v_H - v_L)v_L}{2\alpha M(M - n_H + N)},\end{aligned} \tag{35}$$

$$\begin{aligned}p_\phi^* &= \frac{v_L}{2\alpha M(M - n_H + N)^2}\\ &\quad \cdot \{[(M - n_H)^2 + MN](M+N)v_L - n_H^2 N v_H\};\end{aligned} \tag{36}$$

and

$$\pi_{PS}^* = \frac{(M+N)v_L^2}{4\alpha} + \frac{n_H^2(M+N)N(v_H - v_L)^2}{4\alpha M(M - n_H + N)^2}. \tag{37}$$

Next, we will compare probabilistic-selling strategy to the benchmarks across the range of R:

1. When $R > 1 + (M - n_H)/N$, both Benchmark 2 and the probabilistic-selling strategy cannot emerge. The possible optimal strategies here are Benchmark 1 and Benchmark 3. Since $v_L < (n_H v_H)/(M+N)$, we can further conclude that Benchmark 1 is the optimal.

2. When $1 < R \le 1 + (M - n_H)/N$, Benchmark 2 cannot emerge, but the probabilistic-selling strategy can emerge. The possible optimal strategies here are Benchmark 1 and Benchmark 3, and the probabilistic-selling strategy. From the condition $1 < R \le 1 + (M - n_H)/N$, we obtain

$$\frac{n_H v_H}{M+N} \le v_L < \frac{n_H v_H}{n_H + N}. \qquad (38)$$

First, we compare Benchmark 1 and Benchmark 3; $\pi^*_{B1} = n_H v_H^2/(4\alpha)$ and $\pi^*_{B3} = (M+N)v_L^2/(4\alpha)$.

Suppose $\pi^*_{B1} > \pi^*_{B3}$, we get $n_H v_H^2/(4\alpha) > (M+N)v_L^2/(4\alpha) \Rightarrow n_H v_H^2 > (M+N)v_L^2$. Since $v_L < n_H v_H/(n_H+N)$ as in (38), as long as we can prove $n_H v_H^2 > (M+N)(n_H v_H/(n_H+N))^2$, the condition $n_H v_H^2 > (M+N)v_L^2$ holds. From $n_H v_H^2 > (M+N)(n_H v_H/(n_H+N))^2$, we obtain that $n_H^2 + n_H N + N^2 > M n_H$, which is true because $N > M > n_H$. We conclude that Benchmark 3 is dominated by Benchmark 1.

Next, we compare probabilistic selling strategy to Benchmark 1; $\pi^*_{B1} = (n_H v_H^2)/(4\alpha)$. Suppose $\pi^*_{PS} > \pi^*_{B1}$, we get

$$\frac{(M+N)v_L^2}{4\alpha} + \left(\frac{n_H^2(M+N)N(v_H - v_L)^2}{4\alpha M(M - n_H + N)}\right)^2 > \frac{n_H v_H^2}{4\alpha}$$
$$\Rightarrow (M+N)v_L^2 + \frac{n_H^2(M+N)N(v_H - v_L)^2}{M(M-n_H+N)^2} > n_H v_H^2.$$

Set $A = (M+N)$, $B = (n_H^2(M+N)N)/(M(M-n_H+N)^2)$, and $C = n_H$; we get

$$A v_L^2 + B(v_H - v_L)^2 > C v_H^2$$
$$\Rightarrow \quad (A+B)v_L^2 - 2Bv_H v_L + (B-C)v_H^2 > 0.$$

Solving the inequality above, we obtain

$$v_L < \frac{B - \sqrt{B^2 - (A+B)(B-C)}}{(A+B)} v_H \quad \text{and}$$
$$v_L > \frac{B + \sqrt{B^2 - (A+B)(B-C)}}{(A+B)} v_H, \qquad (39)$$

where the term inside the square root $B^2 - (A+B)(B-C) = (((M+N)n_H)/(M(M-n_H+N)))[M(M-n_H+N) - n_H N]$ is positive since $M > n_H$.

We next check whether the interval

$$v_L < \frac{B - \sqrt{B^2 - (A+B)(B-C)}}{A+B} v_H \quad \text{and}$$
$$v_L > \frac{B + \sqrt{B^2 - (A+B)(B-C)}}{A+B} v_H$$

is overlapping with the restriction $n_H v_H/(M+N) \le v_L < n_H v_H/(n_H+N)$.

(i) Compare $(B - \sqrt{B^2 - (A+B)(B-C)}/(A+B))v_H$ with $n_H v_H/(M+N)$, $n_H v_H/(M+N) = (C/A)v_H$. Suppose

$$\frac{B - \sqrt{B^2 - (A+B)(B-C)}}{A+B} v_H < \frac{n_H v_H}{M+N}.$$

Then,

$$\frac{B - \sqrt{B^2 - (A+B)(B-C)}}{A+B} < \frac{C}{A}$$
$$\Rightarrow AB - (A+B)C < \sqrt{B^2 - (A+B)(B-C)}$$
$$\Rightarrow -\frac{(M+N)n_H}{M(M-n_H+N)}[M(M-n_H+N) - n_H N]$$
$$< \sqrt{B^2 - (A+B)(B-C)},$$

which is true.

We conclude that $((B - \sqrt{B^2 - (A+B)(B-C)})/(A+B))v_H < n_H v_H/(M+N)$.

(ii) Compare $((B + \sqrt{B^2 - (A+B)(B-C)})/(A+B))v_H$ with $n_H v_H/(n_H+N)$. Suppose

$$\frac{B + \sqrt{B^2 - (A+B)(B-C)}}{A+B} v_H > \frac{n_H v_H}{n_H + N}.$$

Then,

$$(n_H + N)\sqrt{B^2 - (A+B)(B-C)} > n_H A - NB$$
$$\Rightarrow (n_H+N)\sqrt{\frac{(M+N)n_H}{M(M-n_H+N)}[M(M-n_H+N) - n_H N]}$$
$$> (M+N)n_H - \frac{n_H^2(M+N)N^2}{M(M-n_H+N)^2}$$
$$\Rightarrow (n_H+N)^2[M(M-n_H+N) - n_H N]$$
$$> \frac{(M+N)n_H[M(M-n_H+N)^2 - n_H N^2]^2}{M(M-n_H+N)^3}$$
$$\Rightarrow (n_H+N)^2(M-n_H)M(M-n_H+N)^3$$
$$> n_H[M(M-n_H+N)^2 - n_H N^2]^2$$
$$\Rightarrow (n_H+N)^2 M(M-n_H+N)^3$$
$$> n_H[M(M-n_H) + 2MN - N^2]^2(M-n_H).$$

The left-hand side is equal to

$$n_H^2 M(M-n_H)^3 + n_H^2 MN^3 + 3n_H^2 M(M-n_H)^2 N$$
$$+ 3n_H^2 M(M-n_H)N^2 + 2n_H NM(M-n_H)^3 + 2n_H MN^4$$
$$+ 6n_H(M-n_H)^2 MN^2 + 6n_H(M-n_H)MN^3$$
$$+ (M-n_H)^3 MN^2 + MN^5 + 3(M-n_H)^2 MN^3$$
$$+ 3(M-n_H)MN^4.$$

The right-hand side is equal to

$$n_H M^2(M-n_H)^3 + 4n_H(M-n_H)M^2N^2 + n_H(M-n_H)N^4$$
$$+ 4n_H(M-n_H)^2 M^2 N + 2n_H(M-n_H)^2 MN^2$$
$$+ 4n_H(M-n_H)MN^3.$$

From $N > M > n_H$, we obtain the following:

a. Term $(M-n_H)^3 MN^2$ in the left dominates $n_H M^2 \cdot (M-n_H)^3$ in the right.

b. Term $6n_H(M-n_H)MN^3$ in the left dominates $4n_H \cdot (M-n_H)M^2N^2 + 2n_H(M-n_H)^2 MN^2$ in the right.

c. Term $6n_H(M-n_H)^2 MN^2$ in the left dominates $4n_H \cdot (M-n_H)^2 M^2 N$ in the right.

d. Terms $MN^5+3(M-n_H)MN^4$ in the left dominate $4n_H(M-n_H)MN^3$ in the right.

e. Term $2n_HMN^4$ in the left dominates $n_H(M-n_H)N^4$ in the right.

Since the terms in the left, except the terms mentioned above, are positive, we conclude that the left-hand side is large than the right-hand side, which implies that $((B+\sqrt{B^2-(A+B)(B-C)})/(A+B))v_H > n_Hv_H/(n_H+N)$.

Following these results, we find that there is no overlap between the interval $v_L < ((B-\sqrt{B^2-(A+B)(B-C)})/(A+B))v_H$ and $v_L < ((B-\sqrt{B^2-(A+B)(B-C)})/(A+B))v_H$ when $n_Hv_H/(M+N)\le v_L < n_Hv_H/(n_H+N)$. Therefore, the probabilistic-selling strategy is dominated by Benchmark 1. Again, the optimal strategy is Benchmark 1.

3. When $R\le 1$, both Benchmark 2 and probabilistic-selling strategy will emerge. It is clear that Benchmark 1 is dominated by Benchmark 2. From the condition $R\le 1$, we obtain $v_L\ge n_Hv_H/(n_H+N)$.

Without utilizing probabilistic selling strategy, the seller has two options: Benchmark 2 or Benchmark 3.

We find that when

$$v_L \le \frac{\sqrt{n_H(n_H+N)/N}\,v_H}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}},$$

Benchmark 2 is better than Benchmark 3, whereas when $v_L > \sqrt{n_H(n_H+N)/N}\,v_H/\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}$, Benchmark 3 dominates Benchmark 2.

First, when

$$v_L \le \frac{\sqrt{n_H(n_H+N)/N}\,v_H}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}},$$

Benchmark 2 is the relative benchmark.

Comparing probabilistic-selling strategy to Benchmark 2, we gain several interesting insights:

(i) In Equation (33), the quality of the high-quality product is $v_L/(2\alpha)+Nn_H(v_H-v_L)/(2\alpha M(M-n_H+N))$ in probabilistic-selling and is lower than $v_H/(2\alpha)$ the quality of the high-quality product in the benchmark.

(ii) In Equation (34), the quality of the low-quality product is $v_L/(2\alpha)-n_H(v_H-v_L)/(2\alpha(M-n_H+N))$ in probabilistic selling and is higher than $v_L/(2\alpha)-n_H(v_H-v_L)/(2\alpha N)$, the quality of the low-quality product in the benchmark.

(iii) In Equation (35) the price of the high-quality product in probabilistic selling, $n_H^2N(v_H-v_L)^2/(2\alpha M(M-n_H+N)^2)+v_L^2/(2\alpha)+n_HN(v_H-v_L)v_L/(2\alpha M(M-n_H+N))$, is lower than $v_H^2/(2\alpha)-(v_H-v_L)v_L/(2\alpha)+n_H(v_H-v_L)^2/(2\alpha N)$ in the benchmark.

(iv) Here, we check whether using probabilistic selling can increase the seller's profit. Suppose $\pi^*_{PS} > \pi^*_{B2}$, from Equations (27) and (38), we get

$$\begin{aligned}
&\frac{(M+N)v_L^2}{4\alpha}+\frac{n_H^2(M+N)N(v_H-v_L)^2}{4\alpha M(M-n_H+N)^2}\\
&\quad > \frac{n_Hv_H^2}{4\alpha}+\frac{Nv_L^2}{4\alpha}-\frac{n_H(v_H-v_L)v_L}{2\alpha}+\frac{n_H^2(v_H-v_L)^2}{4\alpha N}\\
&\Rightarrow (M-n_H)v_L^2\\
&\quad > \left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right](v_H-v_L)^2.
\end{aligned}$$

We can prove that the coefficient $(Mn_H(M-n_H+N)^2\cdot(n_H+N)-n_H^2(M+N)N^2)/(MN(M-n_H+N)^2)$ on the right-hand side is positive:

$$\begin{aligned}
&\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\\
&=\big(n_H^2[M(M-n_H+N)^2-n_HN^2]+n_H(M-n_H+N)\\
&\qquad\cdot N[M(M-n_H+N)-n_HN]\big)\cdot(MN(M-n_H+N)^2)^{-1}.
\end{aligned}$$

It is positive since $M>n_H$.

Therefore, the solution of the inequality is

$$\begin{aligned}
v_L > {}&\left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}v_H\right)\\
&\cdot\left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}\right.\\
&\qquad\left.+\sqrt{M-n_H}\right)^{-1}.
\end{aligned}$$

Since we can prove that

$$\begin{aligned}
\frac{n_H}{n_H+N} &< \left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}\right)\\
&\quad\cdot\left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}\right.\\
&\qquad\left.+\sqrt{M-n_H}\right)^{-1}\\
&< \frac{\sqrt{n_H(n_H+N)/N}}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}}
\end{aligned}$$

when $N>M>n_H$, the overall condition for probabilistic selling to emerge as the optimum is

$$\begin{aligned}
&\left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}v_H\right)\\
&\quad\cdot\left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}\right.\\
&\qquad\left.+\sqrt{M-n_H}\right)^{-1}\\
&\le v_L \le \frac{\sqrt{(n_H(n_H+N)/N)v_H}}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}}.
\end{aligned}\tag{40}$$

Next, when

$$v_L > \frac{\sqrt{n_H(n_H+N)/N}\,v_H}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}},$$

Benchmark 3 is the relative benchmark. Since

$$\pi^*_{B3} = \frac{M+N)v_L^2}{4\alpha}\quad\text{and}$$

$$\pi^*_{PS} = \frac{(M+N)v_L^2}{4\alpha}+\frac{n_H^2(M+N)N(v_H-v_L)^2}{4\alpha M(M-n_H+N)^2},$$

the probabilistic-selling strategy dominates Benchmark 3 and is the optimal strategy. Moreover, the quality choices predicted in Proposition 3B follow in a straightforward fashion from Equations (33) and (34).

In summary, we conclude that

—when

$$\gamma_1 v_H \le v_L \le \frac{\sqrt{n_H(n_H+N)/N}\,v_H}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}}, \quad \pi^*_{PS} \ge \pi^*_{B2} \ge \pi^*_{B3};$$

—when

$$v_L > \frac{\sqrt{n_H(n_H+N)/N}\,v_H}{\sqrt{n_H(n_H+N)/N}+\sqrt{M-n_H}}, \quad \pi^*_{PS} > \pi^*_{B3} > \pi^*_{B2}.$$

In general, the probabilistic selling strategy is the optimal strategy as long as $v_L > \gamma_1 v_H$.

Here,

$$\gamma_1 = \left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}\right) \cdot \left(\sqrt{\left[\frac{Mn_H(M-n_H+N)^2(n_H+N)-n_H^2(M+N)N^2}{MN(M-n_H+N)^2}\right]}+\sqrt{M-n_H}\right)^{-1}.$$

Proof of Proposition 4. *Surplus in Benchmark* 2. In Benchmark 2, the low-type consumers do not enjoy any surplus. Surplus only comes from high-type consumers and is given as $(V_{HL}-V_{LL})$. Since $V_{HL}=v_H q_L$ and $V_{LL}=v_L q_L$, the overall consumer surplus is

$$S_{B2} = n_H(v_H-v_L)q_L. \tag{41}$$

Surplus in Benchmark 3. In Benchmark 3, the low-type consumers do not enjoy any surplus. Surplus only comes from high-type consumers and is given as $(V_{HH}-V_{LH})$. Since $V_{HH}=v_H q_H$ and $V_{LH}=v_L q_H$, the overall consumer surplus is

$$S_{B3} = n_H(v_H-v_L)q_H = \frac{n_H(v_H-v_L)v_L}{2\alpha}. \tag{42}$$

Surplus in Probabilistic Selling. Under probabilistic selling also, the low-type consumers do not enjoy surplus. The high-type consumer's surplus is

$$\phi(V_{HH}-V_{LH})+(1-\phi)(V_{HL}-V_{LL}).$$

Thus, the overall consumer surplus here is

$$\begin{aligned} S_\phi &= n_H[\phi(V_{HH}-V_{LH})+(1-\phi)(V_{HL}-V_{LL})] \\ &= n_H[(V_{HL}-V_{LL})+\phi\Delta]. \end{aligned}$$

Since $V_{HL}=v_H q_L$ and $V_{LL}=v_L q_L$ and $\Delta=(v_H-v_L)\cdot(q_H-q_L)$, we get

$$\begin{aligned} S_\phi &= n_H(v_H-v_L)q_L + n_H\phi(v_H-v_L)(q_H-q_L) \\ &= \frac{n_H(v_H-v_L)v_L}{2\alpha} - \frac{n_H^3 N(v_H-v_L)^2}{2\alpha M(M-n_H+N)^2}. \end{aligned} \tag{43}$$

Because the quality of the low-quality product in the probabilistic-selling strategy is higher than in Benchmark 2, and the term $n_H\phi(v_H-v_L)(q_H-q_L)$ in S_ϕ is positive, we conclude that the aggregate consumer surplus with probabilistic selling is greater than the consumer surplus in Benchmark 2. But the aggregate consumer surplus with probabilistic selling is lower than the consumer surplus in Benchmark 3 since $S_{B3}=n_H(v_H-v_L)v_L/(2\alpha)$ and $S_\phi = n_H(v_H-v_L)v_L/(2\alpha) - n_H^3 N(v_H-v_L)^2/2\alpha M(M-n_H+N)^2$.

Proof of Proposition 5. *Benchmarks.* Following the discussions presented in the proof of Proposition 2, we obtain the following.

Benchmark 1. Target the high-quality product to the high-type segment only.

The seller's profit is $\pi_{B1} = [\theta n_{\underline{H}} + (1-\theta)M](p_H-c_H)$.

The optimal quality choice is $q_H^* = v_H/(2\alpha)$.

And

$$\pi^*_{B1} = \frac{[\theta n_{\underline{H}} + (1-\theta)M]v_H^2}{4\alpha}. \tag{44}$$

Benchmark 2. Target the high-quality product to the high-type segment and the low-quality product to low-type segment.

The seller's profit is $\pi_{B2} = [\theta n_{\underline{H}} + (1-\theta)M](p_H-c_H) + N(p_L-c_L)$.

The optimal qualities are $q_H^* = v_H/(2\alpha)$, $q_L^* = v_L/(2\alpha) - [\theta n_{\underline{H}} + (1-\theta)M](v_H-v_L)/(2\alpha N)$ and

$$\begin{aligned} \pi^*_{B2} = &\frac{[\theta n_{\underline{H}} + (1-\theta)M]v_H^2}{4\alpha} + \frac{N v_L^2}{4\alpha} \\ &- \frac{[\theta n_{\underline{H}} + (1-\theta)M](v_H-v_L)v_L}{2\alpha} \\ &+ \frac{[\theta n_{\underline{H}} + (1-\theta)M]^2(v_H-v_L)^2}{4\alpha N}. \end{aligned} \tag{45}$$

Since q_L^* is nonnegative, we have $R' = [\theta n_{\underline{H}} + (1-\theta)M]\cdot(v_H-v_L)/(Nv_L) \le 1$ to ensure that Benchmark 2 can emerge. From $R' \le 1$, we obtain

$$v_L \ge \frac{[\theta n_{\underline{H}} + (1-\theta)M]v_H}{[\theta n_{\underline{H}} + (1-\theta)M]+N}. \tag{46}$$

Benchmark 3. Target the high-quality product to both the low- and high-type segments and the low-quality product to the low-type segment.

The seller's profit is $\pi_{B3} = M(p_H-c_H) + N(p_L-c_L)$.

The optimal qualities are $q_H^* = v_L/(2\alpha)$ and $q_L^* = v_L/(2\alpha)$.

And

$$\pi^*_{B3} = \frac{(M+N)v_L^2}{4\alpha}. \tag{47}$$

Probabilistic Selling. The seller faces demand uncertainty with respect to demand from high-type consumers. When the seller utilizes probabilistic selling, he needs to pick Y, the optimal level of high-quality capacity targeted to high-type consumers, and X, the amount of low-quality capacity placed in the synthetic product. Since we set transaction cost $c=0$, the optimal strategy for the seller is to assign all the low-quality capacity to the synthetic product; $X=N$. Therefore, the probability that the buyer of the synthetic product obtains the high-quality product is $\phi=(M-Y)/(M-Y+N)$. As before, the demand of low-type consumer is n_L. We next consider the two demand realizations:

1. When the realized demand is $n_{\underline{H}}$, the seller's profit is (refer to (7))

$$\pi_{\underline{H}}=n_{\underline{H}}(p_H-c_H)+(M-Y+X)[p_\phi-\phi c_H-(1-\phi)c_L]+(N-X)(p_L-c_L), \quad (48)$$

$$\pi_{\underline{H}}=n_{\underline{H}}(p_H-c_H)+(M-Y+N)[p_\phi-\phi c_H-(1-\phi)c_L],$$

since $X=N$.

2. When the realized demand is $n_{\bar{H}}$, the seller's profit is (refer to (7))

$$\pi_{\bar{H}}=Y(p_H-c_H)+(M-Y+X)[p_\phi-\phi c_H-(1-\phi)c_L]+(N-X)(p_L-c_L), \quad (49)$$

$$\pi_{\bar{H}}=Y(p_H-c_H)+(M-Y+N)[p_\phi-\phi c_H-(1-\phi)c_L],$$

since $X=N$; where

$$p_H=(v_Hq_H-v_Hq_L+v_Lq_L)-\frac{(M-Y)\Delta}{M-Y+N},$$

$$p_\phi=\frac{(M-Y)v_Lq_H}{M-Y+N}+\frac{Nv_Lq_L}{M-Y+N}, \quad \text{and} \quad p_L=v_Lq_L.$$

The seller's overall profit is

$$\begin{aligned}\pi &= \theta\pi_{\underline{H}}+(1-\theta)\pi_{\bar{H}}\\ &=\frac{[\theta n_{\underline{H}}+(1-\theta)Y]N(v_H-v_L)(q_H-q_L)}{M-Y+N}+\theta(n_{\underline{H}}-Y)v_Lq_H\\ &\quad+M(v_Lq_H-\alpha q_H^2)+N(v_Lq_L-\alpha q_L^2)\end{aligned} \quad (50)$$

s.t. $n_{\underline{H}}\le Y\le M$.

The first-order condition yields

$$\begin{aligned}\frac{\partial\pi}{\partial Y}&=\frac{(1-\theta)N(v_H-v_L)(q_H-q_L)}{M-Y+N}\\ &\quad+\frac{[\theta n_{\underline{H}}+(1-\theta)Y]N(v_H-v_L)(q_H-q_L)}{(M-Y+X)^2}\\ &\quad-\theta v_Lq_H=0,\end{aligned} \quad (51)$$

$$\begin{aligned}\frac{\partial\pi}{\partial q_H}&=\frac{[\theta n_{\underline{H}}+(1-\theta)Y]N(v_H-v_L)}{M-Y+N}\\ &\quad+\theta(n_{\underline{H}}-Y)v_L+M(v_L-2\alpha q_H)=0,\end{aligned} \quad (52)$$

$$\begin{aligned}\frac{\partial\pi}{\partial q_L}&=-\frac{[\theta n_{\underline{H}}+(1-\theta)Y]N(v_H-v_L)}{M-Y+N}\\ &\quad+N(v_L-2\alpha q_L)=0.\end{aligned} \quad (53)$$

The second-order condition yields

$$\frac{\partial^2\pi}{\partial Y^2}=\frac{2(1-\alpha)X\Delta}{(M-Y+X)^2}+\frac{2[\alpha n_{\underline{H}}+(1-\alpha)Y]X\Delta}{(M-Y+X)^3}>0,$$

$$\begin{aligned}\frac{\partial^2\pi}{\partial Y^2}&=\frac{2(1-\theta)N(v_H-v_L)(q_H-q_L)}{(M-Y+X)^2}\\ &\quad+\frac{2[\theta n_{\underline{H}}+(1-\theta)Y]N(v_H-v_L)(q_H-q_L)}{(M-Y+X)^3}>0,\end{aligned} \quad (54)$$

$$\frac{\partial^2\pi}{\partial q_H^2}=-2\alpha M<0, \quad (55)$$

$$\frac{\partial^2\pi}{\partial q_L^2}=-2\alpha N<0. \quad (56)$$

Since $\partial^2\pi/\partial Y^2>0$, but $\partial^2\pi/\partial q_H^2$ and $\partial^2\pi/\partial q_L^2$ are negative, the solution from the first-order condition is at the saddle point, but not the maximum. The optimal solution is at the corner.

Corner Solution 1: $Y=n_{\underline{H}}$; this is the probabilistic-selling strategy, and the profit is

$$\pi=\frac{n_{\underline{H}}N(v_H-v_L)(q_H-q_L)}{M-n_{\underline{H}}+N}+M(v_Lq_H-\alpha q_H^2)+N(v_Lq_L-\alpha q_L^2).$$

Solving this problem, we get

$$q_H^*=\frac{v_L}{2\alpha}+\frac{Nn_{\underline{H}}(v_H-v_L)}{2\alpha M(M-n_{\underline{H}}+N}\quad\text{and}$$

$$q_L^*=\frac{v_L}{2\alpha}-\frac{n_{\underline{H}}(v_H-v_L)}{2\alpha(M-n_{\underline{H}}+N)};$$

$$\pi^*_{C1(PS)}=\frac{(M+N)v_L^2}{4\alpha}+\frac{n_{\underline{H}}^2(M+N)N(v_H-v_L)^2}{4\alpha M(M-n_{\underline{H}}+N)^2}. \quad (57)$$

Since q_L^* is nonnegative, we need

$$\frac{n_{\underline{H}}(v_H-v_L)}{(M-n_{\underline{H}}+N)v_L}\le 1$$

to ensure probabilistic selling can emerge. It implies that $v_L\ge n_{\underline{H}}v_H/(M+N)$ and $R'=[\theta n_{\underline{H}}+(1-\theta)M](v_H-v_L)/(Nv_L)\le[\theta n_{\underline{H}}+(1-\theta)M](M-n_{\underline{H}}+N)/(n_{\underline{H}}N)$.

Corner Solution 2: $Y=M$; it is indistinguishable from Benchmark 2.

Following the discussion in the proof of Proposition 2, the probabilistic-selling strategy can emerge as a possible optimal strategy only when $R'=[\theta n_{\underline{H}}+(1-\theta)M](v_H-v_L)/(Nv_L)\le 1$, which implies that $v_L\ge[\theta n_{\underline{H}}+(1-\theta)M]v_H/([\theta n_{\underline{H}}+(1-\theta)M]+N)$. And the relative benchmarks in this condition are still Benchmarks 2 and 3. Since Benchmark 3 is unconditionally dominated by probabilistic selling, as stated in Proposition 3A, the condition for the probabilistic selling to be optimal is

$$\begin{aligned}&\pi^*_{C1(PS)}>\pi^*_{B2}\\ &\Rightarrow\frac{(M+N)v_L^2}{4\alpha}+\frac{n_{\underline{H}}^2(M+N)N(v_H-v_L)^2}{4\alpha M(M-n_{\underline{H}}+N)^2}\\ &\quad>\frac{[\theta n_{\underline{H}}+(1-\theta)M]v_H^2}{4\alpha}+\frac{Nv_L^2}{4\alpha}-\frac{[\theta n_{\underline{H}}+(1-\theta)M](v_H-v_L)v_L}{2\alpha}\\ &\quad+\frac{[\theta n_{\underline{H}}+(1-\theta)M]^2(v_H-v_L)^2}{4\alpha N}\\ &\Rightarrow v_L>\Big[\big(M[\theta n_{\underline{H}}+(1-\theta)M](M-n_{\underline{H}}+N)^2[[\theta n_{\underline{H}}+(1-\theta)M]\\ &\qquad+N]-n_{\underline{H}}^2(M+N)N^2\big)\cdot(MN(M-n_{\underline{H}}+N)^2)^{-1}\Big]^{1/2}v_H\\ &\qquad\cdot\Big\{\Big[\big(M[\theta n_{\underline{H}}+(1-\theta)M](M-n_{\underline{H}}+N)^2\\ &\qquad\cdot[[\theta n_{\underline{H}}+(1-\theta)M]+N]-n_{\underline{H}}^2(M+N)N^2\big)\\ &\qquad\cdot(MN(M-n_{\underline{H}}+N)^2)^{-1}\Big]^{1/2}\\ &\qquad+\sqrt{M-[\theta n_{\underline{H}}+(1-\theta)M]}\Big\}^{-1}.\end{aligned}$$

Since we can prove that $[\theta n_{\underline{H}}+(1-\theta)M]v_H/([\theta n_{\underline{H}}+(1-\theta)M]+N)<\gamma_2 v_H$, therefore, the overall condition for probabilistic selling to emerge as optimum is $v_L\ge\gamma_2 v_H$.

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